Chapter 6

Large-scale waves and transport in middle latitudes

6.1 Vertical propagation of planetary waves

We consider the properties of small-amplitude quasigeostrophic waves on a basic state consisting of static stability \( S(z) \), zonal wind \( U(z) \), potential vorticity \( Q(y, z) = y \frac{\partial Q}{\partial y}(z) \) and in which the meridional circulation is zero (to leading order, \( i.e. \), it would be zero if the waves were not there).

The linearized QGPV equation is

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) q' + v' \frac{\partial Q}{\partial y} = \mathcal{X}'
\]

where

\[
\frac{\partial Q}{\partial y} = \left( \frac{\partial}{\partial z} \right) \left( \frac{\partial f_0}{\partial z} N^2 \frac{\partial}{\partial z} \right)
\]

(since \( U = U(z) \)). We assume, for now, conservative flow so that \( \mathcal{X}' = 0 \).

\[
q' = \Delta^2 \psi' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho f_0}{N^2} \frac{\partial \psi'}{\partial z} \right)
\]

and so

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho f_0}{N^2} \frac{\partial \psi'}{\partial z} \right) \right) + \frac{\partial \psi'}{\partial x} \frac{\partial Q}{\partial y} = 0.
\]
We look for solutions of the form

$$\psi' = \Re e^{ik(x-ct)+iy},$$

where $\Psi(z)$ is to be determined. Then we have

$$(U-c) \left( \frac{1}{\rho} \frac{d}{dz} \left( \frac{\rho f_0 \, d\Psi}{N^2 \, dz} \right) - \kappa^2 \Psi \right) + \frac{\partial Q}{\partial y} \Psi = 0.$$ 

Now, if $U$ and $N^2$ are independent of height, this equation may have wavelike solutions $\Psi \approx e^{imz+z/2H}$ (recall that $\rho = \rho_0 e^{-z/H}$). In general, when $U = U(z)$ and $S = S(z)$ the solution is analytically intractable. However, we can still find tractable solutions if $U$ and $S$ are not quite constant, i.e., if they vary only slowly with height. Specifically, we look for solutions of the form

$$\Psi = \Psi_0 \, e^{i\phi(z)} \equiv \Psi_0 \exp \left( i \int^z m(z')dz' \right)$$

where $\Psi_0$ and $m$ are slowly-varying functions of height (in a sense to be defined below); $m$ (which may be complex) is the local wavenumber of the disturbance. We assume that the height scale of the disturbance, $m^{-1}$ (the scale on which the phase, $\alpha$, varies) is formally comparable with the scale height $H$, but that the basic state $U$, $S$ and $\partial Q/\partial y$ vary over a much larger height scale $\hat{H} \gg m^{-1}$. [This approach is known as the WKB approximation; discussion of the extent to which it is reasonable in cases of interest will be discussed later.] Since $m$ will be a function of $U$, $S$ and $\partial Q/\partial y$, we anticipate that it will also vary on the height scale $\hat{H}$; we must also allow $\Psi_0$ to vary on this scale.

Now,

$$\frac{d\Psi}{dz} = \left( im \, \Psi_0 + \frac{d\Psi_0}{dz} \right) e^{i\phi}$$

and

$$\frac{d^2\Psi}{dz^2} = \left( -m^2\Psi_0 + i \frac{dm}{dz} \Psi_0 + 2im \frac{d\Psi_0}{dz} + \frac{d^2\Psi_0}{dz^2} \right) e^{i\phi}.$$ 

The ratio of the second to the first term on the RHS is $\frac{dm}{dz} m^{-1} \approx (m\hat{H})^{-1}$ and of the third to the first is also $(m\hat{H})^{-1}$; of the fourth to the first, $(m\hat{H})^{-2}$; these are small by our assumptions. To leading order, therefore, we write

$$\frac{d^2\Psi}{dz^2} \approx -m^2\Psi_0 \, e^{i\phi}.$$
We also note that
\begin{equation}
\frac{d}{dz} \left( \frac{1}{N^2} \frac{d \Psi}{dz} \right) \approx \frac{1}{N^2} \frac{d^2 \Psi}{dz^2},
\end{equation}
for the same reasons. Then our wave equation becomes
\begin{equation}
m^2 + i \frac{m}{H} - \frac{N^2}{f_0^2} \left( \frac{\partial Q}{\partial y} - \kappa^2 \frac{U}{(U - c)} \right) = 0,
\end{equation}
where \(\kappa^2 = k^2 + \ell^2\) is the square of horizontal wavenumber.

We make life a little simpler by writing
\begin{equation}
m = M - i/2H
\end{equation}
when the wave solution is
\begin{equation}
\Psi = \Psi_0 e^{z/2H} \exp \left( i \int^z (z') \, dz' \right)
\end{equation}
and we have
\begin{equation}
M^2 = \frac{N^2}{f_0^2} \left( \frac{\partial Q}{\partial y} - \kappa^2 \frac{U}{(U - c)} \right) - \frac{1}{4H^2}.
\end{equation}
Alternatively, we may write this as
\begin{equation}
c - U = -\frac{\partial Q}{\partial y} \left( \kappa^2 + \frac{f_0^2}{N^2} M^2 + \frac{f_0^2}{4N^2H^2} \right)^{-1}.
\end{equation}
This is the dispersion relation for three-dimensional Rossby waves of given \(k, \ell, c\) in a stratified atmosphere. If the r.h.s is negative, \(M\) is pure imaginary, so that \(\Psi\) simply grows or decays with height without change of phase. If \(M = \pm i\mu\),
\begin{equation}
\Psi = \Psi_0 e^{z/2H} \exp \left( - \int^z \pm \mu(z') \, dz' \right).
\end{equation}
The kinetic energy density of the wave is
\begin{equation}
\frac{1}{2} \rho (\overline{u^2 + v^2}) = \frac{1}{4} \rho_k^2 |\Psi|^2 = \frac{1}{4} \rho_0 k^2 |\Psi_0|^2 \exp \left( -2 \int^z \pm \mu(z') \, dz' \right).
\end{equation}
We thus ensure boundedness as \(z \to \infty\) by choosing the positive sign for \(\mu\).

If the r.h.s is positive, \(M\) is real and so we have vertically-propagating waves. These may be upward or downward-propagating, depending on the sign of \(M\). The group velocity in the meridional plane is \(c_g = (c_{gy}, c_{gz})\) where
\( c_{gy} = k \frac{\partial c}{\partial \ell} = 2kl \Lambda ; \ c_{gz} = k \frac{\partial c}{\partial M} = 2 \frac{f_s^2}{N^2} kM \Lambda \),

where
\[
\Lambda = \frac{(U - c)^2}{\partial Q/\partial y}.
\]

If we define \( k > 0 \) (which we are free to do without loss of generality), then, if \( \partial Q/\partial y > 0 \), we see that \( M > 0 \) implies upward propagation, and \( M < 0 \) downward propagation.

Note also that even if \( M \) is real and the basic state is uniform \( (U, N^2, \partial Q/\partial y \) constant), \( \psi' \) varies not simply as \( e^{iMz} \), but as
\[
\psi' \simeq e^{iMz} e^{z/2H}
\]

or
\[
|\psi'|^2 \simeq e^{z/H} \simeq \rho_0 / \rho.
\]

In the atmosphere, the amplitude of a vertically-propagating wave \( (M \) real) increases with height inversely as the square root of pressure. For a wave originating at the surface (1000 hPa) and propagating up to the stratopause (1 hPa), its amplitude would (other things being equal — we shall see later that they are not) increase thirtyfold. This must be the case in the absence of dissipation, as the EP theorem shows; we will address this below.

### 6.2 Wave propagation into the stratosphere and mesosphere

The Rossby wave dispersion relation gives us a lot of insight into the large-scale wave climatology of the middle atmosphere and, in particular, why much of the tropospheric wave activity is not evident there. Waves with \( M^2 > 0 \) will propagate readily upward and (to the extent that our conservative model is a realistic one) increase in amplitude with height. Those with \( M^2 < 0 \), on the other hand, will either increase in amplitude with height more slowly or, if \( M^2 < -1/4H^2 \), decrease with height and will therefore be weak relative to the propagating waves. We can easily see that \( M^2 \) will be positive if the “Charney-Drazin” condition:
\[
0 < U - c < U_c,
\]
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is satisfied where $U_c$—the “Rossby critical velocity”—is

$$U_c = \frac{\partial Q}{\partial y} \left( \kappa^2 + \frac{f_0^2 H^2}{4N^2} \right)^{-1}.$$ 

There is thus a propagation “window” for the mean winds. Note that $U_c$ decreases with increasing $\kappa$, so the window becomes narrow for small-scale waves. An example is shown—for constant $U$ (so $\frac{\partial Q}{\partial y} = \beta$)—in Fig 6.2, using the values $H = 7km$, $N^2 = 4 \times 10^{-4}s^{-2}$ and $f_0 = 10^{-4}s^{-1}$ and $\beta = 1.6 \times 10^{-11} m^{-1}s^{-1}$.

For a synoptic scale wave with, say, $k = \ell = \pi/(1000km)$ whence $\kappa^2 = 1.96 \times 10^{-11} m^{-2}$, the propagation window is tiny (about 1$m s^{-1}$ wide); since $U$ in fact varies considerably with height a wave of such scale must be evanescent almost everywhere, if not everywhere. In fact if $U - c \geq 10ms^{-1}$, $\mu$ turns out to be about $(1km)^{-1}$ so the wave amplitude decays much more rapidly than $e^{-z/2H}$; clearly we do not expect to see such waves in the middle atmosphere, except just above the tropopause. This, then, explains the absence of tropospherically-forced, synoptic-scale eddies in the stratosphere and mesosphere.

The window becomes wider for larger-scale waves. For the very largest planetary scale wave with 10000km half-wavelength in the $x$ and $y$ directions, $U_c$ is about 60$m s^{-1}$, which means it is very much easier for these waves to propagate. Note, however, that $U - c$ must be positive; for the quasistationary waves with $c \approx 0$, this means that no waves can propagate through easterlies.
Since we saw at the beginning that stratospheric winds are easterly in summer, we thus have an explanation for the absence of large-scale wave activity in summer. This is shown clearly in the observed wave seasonal cycle in Fig ?? . During northern hemisphere winter, planetary waves (of only the very large scales) can propagate through the westerlies (or at least through enough of them to reach and sustain large amplitudes). There is a notable difference between the northern and southern hemispheres, however. While waves are present right through the northern winter (albeit with large amplitude fluctuations) there is a double-peaked structure to the southern hemisphere cycle, with a midwinter minimum. This is not yet fully explained, though it seems likely that the reason for this minimum lies in the fact that the midwinter westerlies are, as we saw, stronger than in the northern hemisphere. This may mean that \( U > U_c \) and therefore propagation is forbidden because the westerlies are simply too strong. [This simple analysis is not accurate enough for us to be definite on the numerical values of \( U_c \), so it is not as easy as it may seem to resolve this question. In fact, when allowance is made for spherical geometry, it is not easy to understand the reasons for the peak of wave activity in early winter.]

### 6.3 Eliassen-Palm fluxes

Now, \( (F_y, F_z) = \left( \frac{\partial \psi'}{\partial x}, \frac{\partial \psi'}{\partial y}, \frac{\rho f_0}{N^2} \frac{\partial \psi'}{\partial z} \right) \).

Using our expression for \( \psi' \) in the propagating regime (real \( M \)), we have

\[
\psi' = R \epsilon \Gamma(z)e^{i\varphi},
\]

where \( \Gamma = \Psi_0 e^{z/2H} \) and \( \varphi = \int^z M(z')dz' \). So

\[
\frac{\partial \psi'}{\partial x} = R \epsilon ik\ e^{i\varphi},
\]

\[
\frac{\partial \psi'}{\partial y} = R \epsilon il\ e^{i\varphi},
\]

\[
\frac{\partial \psi'}{\partial z} = R \epsilon (iM + \frac{1}{2H})\Gamma e^{i\varphi},
\]

whence
F_y = \frac{1}{2} \rho \, k \ell |\Gamma|^2 = \frac{1}{2} \rho_0 \, k \ell |\Psi_0|^2,

and

F_z = \frac{1}{2} \rho f_0 \, k M |\Gamma^2| = \frac{1}{2} \rho_0 \, k M \frac{f_0^2}{N^2} |\Psi_0|^2.

[Since $F_y$ is independent of $y$, $\nabla \cdot F = \partial F_z / \partial z = 0$ in this case, from the EP theorem. If, therefore, $U$ and/or $N^2$ vary with height in our WKB sense, this theorem tells us how $|\Psi_0|$ varies with height without us having to plough through the mathematics of the next order of approximation. Since $M = M(z)$ then, $|\Psi_0|^2$ must vary as $N^2 / M$. Note that in the case where $U$ and $N^2$ are constant, we can interpret the $e^{z/2H}$ growth of amplitude with height as a manifestation of constant EP flux in the presence of decreasing $\rho$].

Now, the wave activity density is, under our WKB assumptions,

$A = \frac{\rho \cdot \overline{q'^2}}{2 \partial Q / \partial y}$

and

$q' \approx -\left( \kappa^2 + \frac{f_0^2}{N^2} (M^2 + \frac{1}{4H^2}) \right) \Re \Psi_0 e^{i \varphi} e^{z/2H},$

whence

$A \approx \frac{\rho_0}{4A} |\Psi_0|^2.$

Therefore

$F = (F_y, F_z) = \left( 2k \ell \Lambda, 2 \frac{f_0^2}{N^2} k M \Lambda \right) A = (c_{gy}, c_{gz}) A ;$

as we noted (but did not prove) before, the EP flux is equal to the wave activity density times group velocity.

For an upward propagating wave with $k > 0 (c_{gz} > 0, M > 0)$, the wave structure in the $(x - z)$ plane is qualitatively as shown in the Fig 6.1. In reality, the growth of atmospheric waves with height is limited by dissipation, by refraction (the waves spread out in latitude and therefore the wave activity is diluted) and, perhaps most of all, nonlinear effects (as we shall see later). Nevertheless, as Fig ?? illustrates, planetary wave amplitudes can become large at high levels. Note the westward tilt with height ($k M > 0$), and the fact that (for the northern hemisphere) the northward velocity fluctuation is
in phase with the temperature fluctuation, \(i.e., \overline{v'T} > 0\); this implies that \(F_z\) is positive, which it must of course be for an upward propagating wave. In general the requirement that \(F_z\) be upward means that an upward (downward) propagating wave necessarily has poleward (equatorward) eddy heat flux.

6.4 Forced quasistationary waves on a \(\beta\)-plane

We consider now the response of a stratified atmosphere to stationary topographic forcing (response to stationary thermal forcing is in most respects similar). As before, we assume that the basic state stratification \(N^2\) and mean zonal wind \(U\) are slowly varying functions of height. We now allow the flow to be nonconservative to the extent that it is dissipated by weak Newtonian cooling

\[ Q = -\alpha \, T, \]

where \(\alpha\) may also be a slowly-varying function of height. Then, in the quasigeostrophic potential vorticity equation

\[ \nabla = -\frac{f_0}{\rho} \frac{\partial \left( \rho \alpha (T - T_e) \right)}{\partial z}. \]
The β-plane is bounded by rigid zonal walls at \( y = 0, L \). At the lower boundary, we specify a topography \( h_B(x, y) = \hat{h}_0 \cos kx \sin \ell y \), where \( \ell = \pi/L \) and where \( \hat{h}_0/H \) is small. Since the lower boundary is an impermeable material surface then, if we restrict attention to the steady solution

\[ w = u \cdot \nabla h_B \]

where \( h_B \) is the log-pressure height of the topography. Since \( z \approx z_{gT}/T_* \), we have

\[ h_B = \hat{h}_0 \frac{T}{T_*} = \text{Re} \ h_0 e^{ikx} \sin \ell y \]

where \( h_0 = T\hat{h}_0/T_* \). [N.B. we have been able to bypass some subtleties here because we are considering steady flow; in general, topographic boundary conditions in pressure coordinates are tricky because of possible nonzero \( \partial p/\partial t \) on the physical surface.] Linearizing this boundary condition, we have

\[ w' = U \frac{\partial h_B}{\partial x} \]

Now, assuming adiabatic flow on the boundary (i.e., \( \alpha = 0 \) there), the linearized steady thermodynamic equation gives us

\[ U \frac{\partial T'}{\partial x} + w'S = 0 \]

there, whence \( T' = -S h_B \), and so

\[ \frac{\partial \psi'}{\partial z} = -\frac{N^2 h_B}{f_0} \]

on \( z = 0 \). Since \( \psi' = \phi'/f_0 \), if we write

\[ \psi' = \text{Re} \ \Psi(z)e^{ikx} \sin \ell y \]

then, on \( z = 0 \),

\[ \frac{d\Psi}{dz} = -\frac{N^2 h_0}{f_0} \]

Now, our linearized QGPV equation is

\[ U \frac{\partial q'}{\partial x} + \frac{\partial \psi'}{\partial x} \frac{\partial Q}{\partial y} = \chi' \].
We assume that the Newtonian cooling rate $\alpha$ is so small that $\chi'$ is negligible at leading order in our WKB expansion (we shall include it at next order and will discover later how small it has to be for this approximation to be valid). Then
\[
U \left( -\kappa^2 \Psi + \frac{1}{\rho} \frac{f_0^2}{N^2} \frac{d}{dz} \left( \rho \frac{d\Psi}{dz} \right) \right) + \Psi \frac{\partial Q}{\partial y} = 0 ,
\]
whence
\[
\frac{1}{\rho} \frac{d}{dz} \left( \rho \frac{d\Psi}{dz} \right) + \frac{N^2}{f_0^2} \left( \frac{\partial Q/\partial y}{U} - \kappa^2 \right) \Psi = 0 .
\]

We shall assume that the Charney-Drazin criterion $0 < U < U_c$ is satisfied, so that we have propagating solutions of the form
\[
\Psi = \Psi_0(z) e^{z/2H} \exp \left[ i \int_0^z M(z') \, dz' \right] ,
\]
where $\Psi_0(z)$—which is real—and $M(z)$ may be slowly-varying with $z$ and

\[
M^2(z) = \frac{N^2}{f_0^2} \left( \frac{\partial Q/\partial y}{U} - \kappa^2 - \frac{f_0^2}{4N^2H^2} \right) .
\]

The wavenumber $M$ is real and positive (since we demand that our solution be upward-propagating).

To determine $\Psi_0(z)$, we can avoid going to next order in the expansion by using the EP relation
\[
\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = \mathcal{D} ,
\]
where
\[
\mathcal{D} = \rho q' \mathbf{X}' / \frac{\partial Q}{\partial y} .
\]

Using our WKB expressions for $A$ and $\mathbf{F}$, we have
\[
A \approx \frac{p_0}{4\Lambda} |\Psi_0|^2 \sin^2 \ell y
\]
and
\[
\mathbf{F} = (F_y, F_z) = \left( 0, \frac{f_0^2}{N^2} kM \Lambda \right) A = (0, c_{gy}) A ,
\]
where $\Lambda = U^2 \frac{\partial Q}{\partial y}$. [Note that, unlike the previous case with $\psi' \approx e^{i \beta y}$, we have no systematic propagation in $y$ and therefore no $F_y$.] Similarly, it is easy to show to leading order that

\[ \chi' = -\frac{f_0}{\rho} \frac{\partial}{\partial z} \left( \rho a \frac{T'}{S} \right) \]

\[ \simeq \alpha \frac{f_0^2}{N^2} (M^2 + \frac{1}{4H^2}) \text{Re} \Psi_0 z^2/2H \exp \left[ i \int_{z'}^{z} M(z') \, dz' \right] e^{ikz} \sin \ell y \]

Therefore, since $q' = -\psi' \frac{\partial Q}{\partial y}/U$,

\[ D = -\frac{\alpha f_0^2}{2UN^2} \left( M^2 + \frac{1}{4H^2} \right) |\Psi_0|^2 \sin^2 \ell y = -\nu A, \]

thus defining $\nu$, where

\[ \nu = \frac{2\alpha f_0^2 U}{N^2 \frac{\partial Q}{\partial y}} \left( M^2 + \frac{1}{4H^2} \right) = 2\alpha \left( 1 - \frac{\kappa^2 U}{\frac{\partial Q}{\partial y}} \right) \]

is the rate of dissipation of wave activity by the radiative damping. The EP relation therefore becomes

\[ \frac{\partial A}{\partial t} + \frac{\partial}{\partial z} (c_{gz} A) = -\nu A. \]

Since our problem is a steady one, $\partial A/\partial t = 0$ and so, since $F_z = c_{gz} A$,

\[ \frac{\partial F_z}{\partial z} = -\frac{\nu}{c_{gz}} F_z. \]

If there no dissipation, $\nu = 0$ and $F_z$ is constant with height (the EP theorem again). If $\nu \neq 0$, then

\[ F_z(z) = F_z(0) \exp \left( -\int_{0}^{z} \mu(z') \, dz' \right), \]

where $\mu$, the rate of attenuation of EP flux with height, is

\[ \mu = \frac{\nu}{c_{gz}} = \frac{(\text{RATE OF DISSIPATION OF WAVE ACTIVITY})}{(\text{VERTICAL GROUP VELOCITY})}. \]
[N.B. our lower boundary condition was derived on the assumption that $\alpha = 0$ on $z = 0$, so we have assumed that $\nu(0) = 0$. This equation allows us to determine $F_z(z)$, given $F_z(0)$ from the lower boundary condition. [In fact it also gives $\Psi(z)$, but as we are more interested in $F_z$, we concentrate on the latter]. On $z = 0$, we had $\partial \psi' / \partial z = -N^2 h_B / f_0$, whence

$$
\Psi_0(0) = -\frac{N^2 h_0}{f_0 (iM + 1/2H)}
$$

and

$$
F_z(0) = \frac{1}{2} \frac{kMN^2 |h_0|^2}{(M^2 + 1/4H^2)} \sin^2 \ell y .
$$

Note that the EP flux $\to 0$ for $M \to 0$; in that case $\Psi(0)$ is $\pi$ out of phase with $h_0$, i.e., the pressure fluctuation along the surface is $\pi$ out of phase with $h_B$ and there is no form drag—and hence no momentum flux away from the boundary (cf. our example at the end of Section 5 relating EP flux to the stress on a material surface). It is also not difficult to show, if the wave is nonpropagating ($M$ imaginary — evanescent solution in $z$) and weakly dissipated, that $F_z = 0$.

### 6.5 Forcing of the mean state

Given the $O(\epsilon)$ solution we are now in a position to determine the $O(\epsilon^2)$ impact of the planetary wave on the background state. The equations we require are (5.25) — (5.28):

$$
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_* = \mathcal{G}_x + \frac{1}{\rho} \nabla \cdot \mathbf{F} ,
$$

$$
\frac{\partial \bar{T}}{\partial t} + \bar{w}_s S = \bar{Q} ,
$$

$$
\frac{\partial \bar{T}}{\partial y} + \frac{1}{\rho} \frac{\partial (\rho \bar{w}_s)}{\partial z} = 0 ,
$$

and

$$
f_0 \frac{\partial \bar{u}}{\partial z} = -\frac{g}{T_s} \frac{\partial \bar{T}}{\partial y} .
$$
The problem is then simply the mathematical one of calculating the solution for \( \partial \bar{u} / \partial t, \partial \bar{T} / \partial t, \bar{v}_s \) and \( \bar{w}_s \). As before, we use the continuity equation to define a streamfunction \( \chi_* \) such that

\[
\bar{v}_s = -\frac{1}{\rho} \frac{\partial (\rho \chi_*)}{\partial z}; \quad \bar{w}_s = \frac{\partial \chi_*}{\partial y}.
\]

We first take \( \partial / \partial t \) of the thermal wind equation

\[
f_0 \frac{\partial^2 \bar{\pi}}{\partial z \partial t} + \frac{g}{T_*} \frac{\partial^2 \bar{T}}{\partial y \partial t} = 0.
\]

We then substitute from the momentum and heat equations and substitute for \( \bar{v}_s \) and \( \bar{w}_s \) to get

\[
N^2 \frac{\partial^2 \chi_*}{\partial y^2} + f_0^2 \frac{\partial}{\partial z} \left( \frac{1}{\rho} \frac{\partial (\rho \chi_*)}{\partial z} \right) = f_0 \frac{\partial}{\partial z} \left( \frac{1}{\rho} \nabla \cdot \mathbf{F} \right) + \frac{g}{T_*} \frac{\partial Q}{\partial y} + f_0 \frac{\partial \bar{G}_x}{\partial z}.
\]

Given \( \nabla \cdot \mathbf{F} \) from our wave solution, together with \( \bar{Q} \) and \( \bar{G}_x \), we can then solve this for \( \chi_* \), given appropriate boundary conditions. In terms of \( \bar{v} \) and \( \bar{w} \), we know that \( \bar{v} = 0 \) on the sidewalls \( y = 0, L \) so that we may define \( \chi = 0 \) there. Since

\[
\chi_* = \chi + \frac{\nu' T'}{S},
\]

it follows (since \( \nu' = 0 \) on the walls) that \( \chi_* = 0 \) there also.

As \( z \to \infty \), we require boundedness. On \( z = 0 \), the situation is a little subtle because of the presence of the topography. Our boundary condition there is

\[
w = \mathbf{u} \cdot \nabla h_B,
\]

whence, to \( O(\epsilon^2) \),

\[
\bar{w} = \mathbf{u}' \cdot \nabla h'_B = \nabla \cdot \mathbf{u}' h'_B = \frac{\partial}{\partial y} (\nu' h'_B)
\]

(recall that the relevant \( \mathbf{u} \) is the geostrophic one). So \( \bar{w} \) is not necessarily zero on the boundary if \( h_B \) is nonzero. The explanation of this is as follows. If \( \nu' h'_B \) is nonzero, there is a nonzero correlation between \( \nu' \) and \( h_B \). As shown in Fig 6.5a, this means there is a net northward (or southward) mass flux in the “valleys” which is not compensated in the “hills” simply because
the corresponding region is underground. If the flow is adiabatic near the ground (something we have already assumed),

\[ D_g T' + w' S = 0. \]

But

\[ w' = D_g h_B, \]

whence

\[ h_B = -T'/S \]

and

\[ \overline{v' h_B'} = -\overline{v'T'/S} \]

which is negative in our case (and would be for any northern hemisphere upward propagating wave with \( F_z > 0 \)). So \( v' \) tends to be northward in the valleys, as shown in the figure. Because we have sidewalls, this mass flux must be a function of \( y \) and there is therefore a mass convergence; the only way this can be balanced is by a mean vertical motion (as shown in Fig 6.5b) — which is precisely what our equation for \( \bar{w} \) is telling us.

Our boundary condition on \( \chi \) therefore becomes

\[ \chi = -\overline{v'T'/S}. \]
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The condition on \(\chi_*\), therefore, is simply

\[ \chi_* = 0 \]

on \(z = 0\). So the condition on the residual circulation on a topographic boundary is simpler than for the conventional mean\(^1\). (Once again, the simple condition on \(\bar{\omega}_*\) in this steady, small-amplitude situation reflects the Lagrangian problem—there can be no net mass flux through the boundary, no matter what its shape is. A material tube lying on the boundary must remain on the boundary; therefore it can have no mean velocity normal to the boundary.)

6.5.1 The “nonacceleration” case

If our planetary wave is undissipated, we have \(\nabla \cdot \mathbf{F} = 0\) everywhere. We have already seen that the \(O(\epsilon^2)\) problem for \(\partial \pi / \partial t, \partial \bar{\theta} / \partial t, \bar{v}_*\) and \(\bar{w}_*\) is independent of the waves in this limit, provided the boundary conditions are also (and we have just seen that this is true). If \(\bar{Q}\) and \(\bar{G}_x\) are zero, then our solution is simply

\[ \frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{T}}{\partial t} = \chi_* = 0 \]

everywhere. The conventional mean streamfunction is nonzero, however, being

\[ \chi = -\bar{v} \bar{T} / S \]

everywhere (and is a function of \(y\) only). So, in conventional terms, the situation looks like Fig 6.5.1, with the heat budget being satisfied by a balance between the “eddy heat flux convergence”, \(-\partial (\bar{v} \bar{T}) / \partial y\), and mean adiabatic heating and cooling, \(-\bar{w} S = S \left( \partial (\bar{v} \bar{T}) / \partial y \right) / S\) everywhere.

\(^1\)This is not a general statement. On a flat, baroclinic boundary (e.g., in the Eady and Charney baroclinic instability problems) the opposite is true: \(\bar{w}\) is zero but \(\bar{\omega}_*\) is not. In this case, the residual and Lagrangian means behave differently.
6.5.2 The nonconservative case — instantaneous response

We found that, in the presence of Newtonian cooling, the EP flux is specified by

\[ F_y = 0 \]

\[ F_z(z) = F_0 \sin^2 \ell y \exp \left( -\int_0^z \mu(z')dz' \right) , \]

where

\[ \mu(z) = \frac{2\alpha}{c_g} \left( 1 - \frac{\kappa^2 U}{\partial Q/\partial y} \right) \]

and

\[ F_0 = \frac{1}{2} \frac{p_0 k M N^2 |h_0|^2}{(M^2 + 1/4H^2)} . \]

In order to solve the problem, we first expand \( F_z \) in a sine series, using

\[ \sin^2 \ell y = \sum_n \gamma_n \sin n\ell y , \]

where

\[ \gamma_n = \frac{2}{L} \int_0^L \sin^2 \ell y \sin n\ell y \, dy = \frac{4}{\pi n} \frac{(\cos n\pi - 1)}{(n^2 - 4)} . \]

These coefficients have the values
It can be seen from this that the $n = 1$ term is dominant, so we proceed (with a useful degree of accuracy) by considering this term only, writing

$$F_z(z) \simeq \gamma_1 F_0 \exp \left( - \int_0^z \mu(z') dz' \right) \sin \ell y$$

whence

$$\nabla \cdot \mathbf{F} = \gamma_1 F_0 \frac{\partial}{\partial z} \exp \left( - \int_0^z \mu(z') dz' \right) \sin \ell y = -\gamma_1 F_0 \mu \exp \left( - \int_0^z \mu(z') dz' \right) \sin \ell y.$$

Note that $\nabla \cdot \mathbf{F}$ is negative — the wave acts as an easterly force on the mean state.

We now need to solve our equation (6.1) for $\chi_*$. We consider the wave driving to be the only forcing of the meridional circulation, i.e., $\bar{Q} = 0$ and $\bar{G}_x = 0$. Our equation now becomes

$$N^2 \frac{\partial^2 \chi}{\partial y^2} + f_0^2 \frac{\partial}{\partial z} \left( \frac{1}{\rho} \frac{\partial}{\partial z} (\rho \chi_*) \right) = -f_0^2 \frac{dR}{dz} \sin \ell y$$

where

$$R(z) = \gamma_1 f_0^{-1} F_0 \rho^{-1} \mu \exp \left( - \int_0^z \mu(z') dz' \right)$$

i.e.,

$$R(z) \sin \ell y \equiv -f_0^{-1} \frac{1}{\rho} \nabla \cdot \mathbf{F}.$$

Note that $R$ is positive when $\nabla \cdot \mathbf{F}$ is negative. Profiles of $F_z(z)$ and $R(z)$ for constant and nonconstant $\mu$ are shown schematically in Fig 6.5.2.

Now, writing

$$\chi_*(y, z) = X(z) \sin \ell y,$$
our equation becomes
\[
\left( \frac{d}{dz} - \frac{1}{H} \right) \frac{dX}{dz} - \frac{N^2}{f_0^2} \ell^2 X = -\frac{dR}{dz}(z)
\]

In general, the solution is not very straightforward. However, since the problem is linear, we can use a Green function approach, writing

\[
R(z) = \int_0^z R(z_0)\delta(z - z_0)dz_0.
\]

Then in general the solution is

\[
X(z) = \int_0^z G(z, z_0)R(z_0)dz_0,
\]

where the Green function \(G(z, z_0)\) for the problem is the solution to

\[
\left( \frac{d}{dz} - \frac{1}{H} \right) \frac{dG}{dz} - \frac{N^2}{f_0^2} \ell^2 G = -\frac{d}{dz} [\delta(z - z_0)].
\]

Thus the Green function is the solution we would get in response to a \(\delta\)-function in \(R\) (and therefore in \(\nabla \cdot F\), \textit{i.e.}, in the convergence of \(F\))
such as is shown in Fig 6.5.2a. This in itself is an interesting case, apart from its value in constructing the solution to the more general problem.

Now, except at \( z = z_0 \),

\[
G \approx e^{\Lambda z},
\]

where

\[
2\Lambda H = 1 \pm \sqrt{1 + \frac{4H^2}{D^2}}
\]

with

\[
D = f_0/N \ell
\]

which is the “Rossby height” for the problem (i.e., the length scale \( \ell^{-1} \) is the Rossby radius based on the height \( D \)). Thus there are two solutions for \( X \), one of which always grows with height and one always decays (so it is always easy to decide which satisfies our boundedness constraint). For \( H \gg D \), \( \Lambda \approx \pm 1/D \); for \( H \ll D \), \( \Lambda \approx 0, -1/H \) (note that the negative root for \( \Lambda \), decaying upward, has a smaller absolute value than the negative root; this asymmetry reflects the \( e^{-z/H} \) dependence of pressure).

Above \( z_0 \), then

\[
G = A e^{\Lambda^-(z-z')} ,
\]

where \( \Lambda^- \) is the negative, bounded root. To make things simple, let us assume that \( \Lambda^+ z_0 \gg 1 \). Then a solution

\[
G = B e^{\Lambda^+(z-z')}
\]

in \( z < z_0 \) will almost satisfy the boundary condition \( G = 0 \) on \( z = 0 \).

We now need matching conditions across \( z_0 \) to close the solution. If we integrate our o.d.e once across \( z_0 \) we get

\[
\lim_{\zeta \to 0} \int_{z' - \zeta}^{z' + \zeta} \left( \frac{d^2 G}{dz^2} - \frac{1}{H} \frac{dG}{dz} - \frac{N^2}{f_0^2} \ell^2 G \right) dz = \lim_{\zeta \to 0} \int_{z' - \zeta}^{z' + \zeta} \frac{\partial}{\partial z} (\delta [z - z_0]) dz
\]

whence

\[
\Delta \left( \frac{dG}{dz} - \frac{1}{H} G \right) = 0
\]

where \( \Delta \) means the increment across \( z_0 \). Integrating a second time, we get the second condition

\[
\Delta G = -1.
\]
These two conditions give us

$$(\Lambda^- - \frac{1}{H})A = (\Lambda^+ - \frac{1}{H})B ,$$

and

$$A - B = -1 .$$

Hence

$$B = \frac{1}{H} - \Lambda^- \Lambda^+ - \Lambda^- > 0$$

and

$$A = -\frac{\Lambda^+ - \frac{1}{H}}{\Lambda^+ - \Lambda^-} < 0 .$$

The resulting circulation (actually the mass streamfunction $p\chi_s$) is shown in Fig 6.5.2. [Note that this is the Green function for $X$; the full solution for

the case illustrated in Fig 6.5.2, with $\frac{\partial F_z}{\partial y} = -F_0 \delta(z - z_c) \sin \ell y$ has $R(z) = [f_0\rho(z_c)]^{-1} F_0 \delta(z - z_c)$ and therefore the actual solution is $F_0 G(z)/[f_0\rho(z_c)]$, rather than $G(z)$ itself.

The residual northward flow corresponding to this solution is

$$\vec{v}_s = \begin{cases} 
-(\Lambda^- - \frac{1}{H})A e^{\Lambda^- (z - z_0)} \sin \ell y, & z > z_0 \\
-(\Lambda^+ - \frac{1}{H})B e^{\Lambda^+ (z - z_0)} \sin \ell y, & z < z_0 
\end{cases}$$
6.5. FORCING OF THE MEAN STATE

i.e.,

\[ \tilde{v}_* = \begin{cases} \frac{1}{D^2} \xi e^{\Lambda^- (z-z_0)} \sin \ell y , & z > z_0 \\ \frac{1}{D^2} \xi e^{\Lambda^+ (z-z_0)} \sin \ell y , & z < z_0 \end{cases} \]

where

\[ \xi = \left( 1 + \frac{4H^2}{D^2} \right)^\frac{1}{2}. \]

Since \( \tilde{F}_0 < 0 \), \( \tilde{v}_* \) is negative (equatorward) both above and below \( z_0 \), as seen on Fig 6.5.2. The return flow is of course in a delta-function at \( z_0 \), where

\[ \tilde{v}_* = -\frac{\partial \chi_*}{\partial z} = (B - A) \delta(z - z_0) \sin \ell y = \delta(z - z_0) \sin \ell y . \]

This is positive (poleward) as shown on Fig 6.5.2.

The mean acceleration may now be determined from

\[ \frac{\partial \tilde{u}}{\partial t} = f_0 \tilde{v}_* + \frac{1}{\rho} \nabla \cdot \mathbf{F} \]

Above and below \( z_0 \), \( \nabla \cdot \mathbf{F} = 0 \) and so \( \partial \tilde{u}/\partial t = f_0 \tilde{v}_* \); we can obtain \( \partial \tilde{u}/\partial t \) trivially from the above relations for \( \tilde{v}_* \). In particular, the acceleration is everywhere negative. Note that, unlike \( \tilde{v}_* \), the acceleration does not have a delta-function at \( z_0 \), since the delta-function in \( f_0 \tilde{v}_* \) is balanced by that in \( \rho^{-1} \nabla \cdot \mathbf{F} \) there. The temperature tendency follows simply from \( \tilde{w}_* \):

\[ \frac{\partial \tilde{T}}{\partial t} = -\tilde{w}_* S. \]

The situation we have is therefore as depicted in Fig 6.5.2:

1. A wave propagates up from \( z = 0 \), undissipated until it reaches \( z = z_0 \), where all the wave activity is entirely dissipated, implying a \( \delta \)-function easterly force acting on the mean flow.

2. This drives a residual circulation. At \( z = z_0 \), this has a \( \delta \)-function poleward flow, and therefore gives a positive Coriolis acceleration, thus tending to oppose the \( \nabla \cdot \mathbf{F} \) forcing there (shades of Lenz’ law). The recirculation extends a height \( |\Lambda^-|^{-1} \approx \min(\infty, D) \) above and \( (\Lambda^+)^{-1} \approx \min(H, D) \) below\(^2\), where the flow is poleward and thus induces a negative (easterly) acceleration. The vertical residual motion induces adiabatic warming or cooling in the pattern shown.

\(^2\)The mass streamfunction \( \rho \chi_* \), which is shown in the figures, has a height scale \( |\Lambda^- - \frac{1}{\tilde{H}}|^{-1} \approx \min(H, D) \) above and \( (\Lambda^+ - \frac{1}{\tilde{H}})^{-1} \approx \min(\infty, D) \) below.
3. The vertical profile of the mean acceleration in mid-channel. The overall effect of the induced circulation is to smooth out the response \((\partial \pi / \partial t)\) relative to the \(\nabla \cdot \mathbf{F}\) forcing. This is what it must do to keep the mean state in thermal wind balance. (and of course the circulation can only redistribute mean momentum—it cannot create or destroy it).

For a more realistic case in which \(\nabla \cdot \mathbf{F}\) has a smooth profile, we need to regard the above solution as a Green’s function, as we noted earlier. The solution to a smooth forcing is a convolution of the forcing with our Green’s function and so is smoothed out in the vertical. Note that if the vertical scale of the forcing is much greater than \(|\Lambda^-|^{-1}\) and \((\Lambda^+)^{-1}\), the smoothing effect of the residual circulation is ineffective.

Finally, the “conventional” picture of the \(\delta\)-function forcing case is shown in Fig 6.5.2. Since \(F_z = 0\) above \(z_0\), \(\nu^2 T' = 0\) there and \(\chi = \chi_s\). Below \(z_0\),

\[
\text{the constant EP flux (truncated to } \sin \ell y) \text{ which corresponds to our Greens function source } R(z) = \delta(z - z_0) \text{ is } F_z = f_0 \rho(z_0) \sin \ell y; \text{ therefore } \rho \nu^2 T'/S \text{ is a constant and, to within our truncation in } y, \text{ is}
\]

\[
\rho \nu^2 T'/S = p(z_0) \sin \ell y.
\]

The mass streamfunction for the conventional mean is therefore

\[
\rho \chi = \rho \chi_s + \rho(z_0) \sin \ell y.
\]
This case is shown in the figure; note that $\bar{v}$ has no singularity at $z_0$. Of course the mean acceleration and mean temperature tendency are exactly the same but our description of what is going on is different. In this case, far below $z_0(|z - z_0| \gg |\Lambda^+|^{-1})$, we get the cancellation between eddy heat flux and $\bar{w}$ we found in the nonacceleration case. Near $z_0$, they no longer balance and we have nonzero $\partial T/\partial t$. The acceleration of the mean flow is all ascribed to the mean meridional motion.

6.5.3 The steady response

The case we have just done assumed $\bar{Q} = 0$. However, we found nonzero $\partial T/\partial t$ driven by the wave transports, so, while it makes sense to solve the above problem as an initial tendency problem, it becomes invalid as soon as the dynamically-induced changes to $T$, driving the mean temperature out of radiative equilibrium, become significant. We would then have to include a nonzero $\bar{Q}$ in the diagnostic equation for $\chi_*$ and solve the resulting time-dependent problem. This is straightforward in principle but too complex in practice for us to do here. In fact, what we are really interested in is the ultimate steady solution — the climatological response to climatological forcing of the kind we have considered. This turns out to be very easy.

In the steady inviscid case (we assume that the only waves or “eddy friction” present are included in our $\nabla \cdot F$), we must have

$$-f_0 \bar{v}_* = \frac{1}{\rho} \nabla \cdot F.$$ 

So if we know $\nabla \cdot F$, we immediately know $\bar{v}_*$—without ever having to solve an elliptic problem; the response in terms of $\bar{v}_*$ is purely local. In order to reach a steady state, the Coriolis effect of the residual circulation must exactly balance the wave forcing. We can then determine $\bar{w}_*$ from continuity, to find

$$\frac{\partial}{\partial z} (\rho \bar{w}_*) = -\rho \frac{\partial \bar{v}}{\partial y} = \frac{1}{f_0} \frac{\partial}{\partial y} (\nabla \cdot F).$$

Given $\rho \bar{w}_* \to 0$ as $z \to \infty$, this defines $\bar{w}_*$ everywhere. In fact, since $\nabla \cdot F = \partial F_z/\partial z$ and this vanishes at large $z$, we integrate to get

$$\rho \bar{w}_* = \frac{1}{f_0} \frac{\partial F_z}{\partial y}.$$
[Note that this does not satisfy our lower boundary condition — we will get back to this in a moment]. Then the thermodynamic equation must satisfy

\[ Q = \bar{w}_* S = \frac{1}{\rho f_0} \frac{\partial F^Z}{\partial y} . \]

The mean state must be pushed out of radiative equilibrium just enough for diabatic heating/cooling to balance the adiabatic cooling/heating associated with \( \bar{w}_* \). Note that we can calculate the diabatic heating from a knowledge of \( \nabla \cdot F \) — evidence that in the quasigeostrophic regime it is the dynamics that drives the diabatic heating and not vice-versa. From a radiative calculation we can then determine \( \bar{T} \); for our Newtonian cooling, it is just

\[ -\alpha (\bar{T} - T_e) = \bar{Q} . \]

The overall picture is shown in Fig 6.5.3.

Consider now the thorny issue of the bottom boundary condition. This seems to be saying that there is something wrong with this solution—and of course there is. We have a nonzero form drag on the topography and hence a nonzero net stress on the atmosphere. The mean momentum of the atmosphere (given our assumptions) will therefore change with time — there
can be no truly steady state unless other process come to act. The only way a steady state can be achieved appears to be as follows:

1. The mean flow near the ground changes at leading order such that the net stress vanishes (in our case, this can be done either by the mean wind at the surface vanishing so that there is no form drag—in which case the wave will disappear too!—or the mean wind at low levels changing such that vertical propagation is not permitted). The kind of steady circulation we have just discussed will, for a steady forcing, be established locally in the stratosphere quite quickly (in fact on a time scale \( \approx 1/\alpha \approx 10 \) days). However, changing the mean flow lower down will take much longer, on two counts— the meridional velocities will be weaker there because of the pressure factor, and, since \( \partial \tau / \partial t \) is proportional to \( \epsilon^2 \), it will take a time proportional to \( \epsilon^{-2} \) (very long for small \( \epsilon \)) to effect a finite change in the mean flow. So, on a finite timescale (and the forcing only exists during the winter season) our steady state may never be reached at low levels and the situation will be as depicted in Fig 6.10.

2. Alternatively, surface friction will come into play. Even though these effects are weak by most standards, the required force required to balance the topographic stress is not usually very large and can likely be achieved by a weak induced flow at the surface.

Note that if we had forced our wave thermally, rather than by topography, there would be no net momentum input to the atmosphere and we would not have run into this problem. Then \( F_z \) would be zero at the ground and we could get a truly steady state, with the residual circulation being closed off within the forcing region where \( \mathbf{F} \) is divergent (in fact where the effective force on the mean flow, \( \rho^{-1} \nabla \cdot \mathbf{F} \), must exactly balance that in the dissipation region. See Fig 6.5.3. This simple picture assumes that the region of wave dissipation lies directly above the forcing. If the waves were to propagate horizontally as well as vertically, this would not be true and the considerations noted in (i) and (ii) above would become relevant to this case also.
6.6 Spherical geometry

6.6.1 Basic equations

Basic equations for log-pressure coordinates in spherical geometry (under “primitive equation” assumptions of a thin atmosphere):

\[
\frac{Du}{Dt} - \frac{1}{a} uv \tan \varphi - 2\Omega v \sin \varphi = - \frac{1}{a \cos \varphi} \frac{\partial \Phi}{\partial \lambda} + G_{\lambda},
\]

\[
\frac{Dv}{Dt} + \frac{1}{a} u^2 \tan \varphi + 2\Omega u \sin \varphi = - \frac{1}{a} \frac{\partial \Phi}{\partial \varphi} + G_{\varphi},
\]

\[
\frac{DT}{Dt} + \kappa \frac{w}{H} = Q,
\]

where (\(\lambda, \varphi\)) are longitude and latitude, \(a\) and \(\Omega\) the Earth radius and rotation rate and where

\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} + w \frac{\partial}{\partial z}.
\]

The hydrostatic and continuity equations are as before:

\[
\frac{\partial \Phi}{\partial z} = g \frac{T}{T_s},
\]
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and

\[ \nabla \cdot \rho \mathbf{u} = 0 , \]

where, for any vector \( \mathbf{U} = (U, V, W) \),

\[ \nabla \cdot \mathbf{U} \equiv \frac{1}{a \cos \varphi} \frac{\partial U}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial V}{\partial \varphi} (V \cos \varphi) + \frac{\partial W}{\partial z} . \]

6.6.2 Quasigeostrophic equations

We need a set of equations under quasigeostrophic scaling valid over most of the sphere (quasigeostrophic equations are never going to be valid very close to the equator). As a small parameter for the scaling, we choose

\[ \epsilon = L/a \]

where \( L \) is the latitudinal length scale of the motion, and assume \( \epsilon \ll 1 \). (In fact all trigonometric functions of \( \varphi \)—including the Coriolis parameter—are then slowly-varying on the scale \( L/a \)). We further assume that

\[ U/\Omega L \leq O(\epsilon) , \]

where \( U \) is a horizontal velocity scale, so that \( \epsilon \) has the character of a Rossby number. We assume that time-dependence is slow:

\[ (\Omega \tau)^{-1} \leq O(\epsilon) , \]

where \( \tau \) is a characteristic time-scale. We also, as usual, assume that the isentropic slope is small:

\[ |\nabla_h \theta|/|\frac{\partial \theta}{\partial z}| \leq O(\epsilon) \rightarrow |\nabla_h T|/S \leq O(\epsilon) . \]

In fact, this now means that \( S \) may be not only a function of \( z \) (as usual) but also may be a slowly-varying function of horizontal position (\textit{i.e.}, it may vary on the planetary scale). Finally, we assume that frictional effects are weak at leading order.

The leading order terms in the horizontal momentum equations then become

\[ 2\Omega v \sin \varphi = \frac{1}{a \cos \varphi} \frac{\partial \Phi}{\partial \lambda} , \]

\[ 2\Omega u \sin \varphi = -\frac{1}{a} \frac{\partial \Phi}{\partial \varphi} . \]
It is tempting (and common) to use these equations as definitions of the geostrophic velocities. However this causes problems at next order in \( \epsilon \), since then
\[
\nabla \cdot \mathbf{u} = -\frac{1}{2\Omega a^2} \sin^2 \varphi \frac{\partial \Phi}{\partial \lambda} = -\frac{1}{a} \cot \varphi \, v.
\]

So the geostrophic wind is not nondivergent, though the divergence is ageostrophic (\textit{i.e.}, it is \( O(v/a) \approx \epsilon v/L \)). To proceed in this way one has to remember to add a term \(-a^{-1} \cot \varphi \, v\) to the continuity equation for the ageostrophic wind—something that is messy and inconvenient (and is not always remembered). An alternative approach is to \textit{insist} at the outset on a definition of geostrophic wind that is exactly nondivergent. One way of doing this is to replace the Coriolis parameter in the geostrophic equations with \( 2\Omega \sin \varphi_0 \), where \( \varphi_0 \) is a representative latitude. But then the balance is only valid in the immediate vicinity of \( \varphi_0 \)—in which case one might as well use a beta-plane approach.

Another way of defining a nondivergent geostrophic velocity is as follows. In the full equations of motion, we use the identity

\[
\sin \varphi \frac{\partial}{\partial \varphi} \left( \frac{\Phi}{\sin \varphi} \right) = \frac{\partial \Phi}{\partial \varphi} - \Phi \cot \varphi,
\]

where we note that the second term is \( O(\epsilon) \) smaller than the first [since \( \partial \Phi / \partial \varphi \approx \frac{2}{L} \Phi \)]. Therefore we may rewrite the equations of motion as

\[
\frac{Du}{Dt} - \frac{1}{a} \omega \tan \varphi - 2\Omega v \sin \varphi = -\frac{\tan \varphi}{a} \frac{\partial}{\partial \lambda} \left( \frac{\Phi}{\sin \varphi} \right) + G_{\lambda},
\]
\[
\frac{Dv}{Dt} + \frac{1}{a} u^2 \tan \varphi + 2\Omega u \sin \varphi = -\frac{\sin \varphi}{a} \frac{\partial}{\partial \varphi} \left( \frac{\Phi}{\sin \varphi} \right) - \frac{\cot \varphi}{a} \Phi + G_{\varphi}.
\]

Then our geostrophic equations are naturally written

\[
v = \frac{1}{a \cos \varphi} \frac{\partial \psi}{\partial \lambda}; \quad u = -\frac{1}{a} \frac{\partial \psi}{\partial \varphi},
\]

where

\[
\psi = \frac{\Phi}{2\Omega \sin \varphi}.
\]

We have thus arrived at a definition of an exactly nondivergent geostrophic velocity, for which \( \psi \) is the geostrophic streamfunction. The price we pay
for this is an extra term in the \( \nu \)-momentum equation (which in my opinion is preferable to having a corresponding term in the continuity equation—it is a matter of convenience and taste). The hydrostatic balance equation is simply

\[
\frac{\partial \psi}{\partial z} = \frac{g}{2\Omega \sin \varphi} \frac{T}{T}. 
\]

The quasigeostrophic equations (to next order in \( \epsilon \)) now become

\[
D_g u - \frac{1}{a} \nu \tan \varphi - 2 \Omega v_a \sin \varphi = G_\lambda ,
\]

\[
D_g v + \frac{1}{a} \nu^2 \tan \varphi + 2 \Omega u_a \sin \varphi = -\frac{2 \Omega \psi}{a} \cos \varphi + G_\varphi ,
\]

\[
D_g T + w_a S = Q .
\]

where \( D_g \) is the time derivative following the geostrophic flow

\[
D_g \equiv \frac{\partial}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} ,
\]

and the ageostrophic continuity equation is

\[
\nabla \cdot \rho u_a = 0 .
\]

### 6.6.3 QGPV equation

In spherical geometry vorticity is defined by

\[
\zeta = \frac{1}{a \cos \varphi} \left( \frac{\partial \psi}{\partial \varphi} - \frac{\partial}{\partial \lambda} (u \cos \varphi) \right) = \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 \psi}{\partial \varphi^2} + \frac{1}{a^2} \cos \varphi \frac{\partial}{\partial \varphi} \left( \cos \varphi \frac{\partial \psi}{\partial \varphi} \right) ,
\]

and potential vorticity by

\[
q = 2 \Omega \sin \varphi + \Delta \psi ,
\]

where

\[
\Delta \equiv \frac{1}{a^2} \cos^2 \varphi \frac{\partial^2}{\partial \lambda^2} + \frac{1}{a^2} \cos \varphi \frac{\partial}{\partial \varphi} \left( \cos \varphi \frac{\partial}{\partial \varphi} \right) + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{4 \Omega^2}{N^2} \sin^2 \varphi \frac{\partial}{\partial z} \right) .
\]

We form the vorticity equation by taking the curl of the equations of motion as usual. We use the identities

\[
D_g \left( \frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right) = \frac{1}{\cos \varphi} \left( \frac{\partial}{\partial \varphi} \left[ \cos \varphi \{ D_g u - \frac{\nu}{a} \tan \varphi \} \right] - \tan \varphi \frac{\partial}{\partial \lambda} (a^2) \right)
\]
to obtain the result
\[ D_g \left( \frac{1}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right) = \frac{1}{\cos \varphi} \frac{\partial}{\partial \lambda} [D_g v]. \]

\[ D_g \zeta + \frac{2 \Omega v}{a} \cos \varphi + \frac{2 \Omega}{a} \cos \varphi \left( \frac{\partial}{\partial \lambda} (u_a \sin \varphi) + \frac{\partial}{\partial \varphi} (v_a \sin \varphi \cos \varphi) \right) \]
\[ = \frac{1}{a \cos \varphi} \left( \frac{\partial G_{\varphi}}{\partial \lambda} - \frac{\partial}{\partial \varphi} (G_{\lambda} \cos \varphi) \right) \]

Note that the third term on the lhs is not (as in the Cartesian case) the horizontal divergence of \( u_a \), but of \( (u_a \sin \varphi) \)—the \( \sin \varphi \) comes from the Coriolis term. Therefore this term is
\[ 2 \Omega \sin \varphi \left( \frac{\partial u_a}{\partial \lambda} + \frac{\partial}{\partial \varphi} (v_a \cos \varphi) \right) + \frac{2 \Omega v_a}{a} \cos \varphi = \frac{2 \Omega v_a}{a} \cos \varphi - \frac{2 \Omega}{\rho} \sin \varphi \frac{\partial}{\partial z} (\rho w_a). \]

Therefore the vorticity equation becomes
\[ D_g \zeta + \frac{2 \Omega v}{a} \cos \varphi + \frac{2 \Omega v_a}{a} \cos \varphi \frac{2 \Omega}{\rho} \sin \varphi \frac{\partial}{\partial z} (\rho w_a) \]
\[ = \frac{1}{a \cos \varphi} \left( \frac{\partial G_{\varphi}}{\partial \lambda} - \frac{\partial}{\partial \varphi} (G_{\lambda} \cos \varphi) \right) \]

The extra term we have gained—the advection of planetary vorticity by the ageostrophic wind—is formally small (being \( \mathcal{O}(\epsilon) \) smaller than the second term) and so we may neglect it. [This term was, in part, retained by Matsuno (1970) and so his final result is slightly different from ours.] Therefore the vorticity equation becomes
\[ D_g (\zeta + 2 \Omega \sin \varphi) - \frac{2 \Omega \sin \varphi}{\rho} \frac{\partial}{\partial z} (\rho w_a) = \frac{1}{a \cos \varphi} \left( \frac{\partial G}{\partial \lambda} \varphi - \frac{\partial}{\partial \varphi} (G_{\lambda} \cos \varphi) \right). \]

Since
\[ w_a = \frac{1}{S} (Q - D_g T) \approx \frac{Q}{S} - D_g \left( \frac{T}{S} \right), \]

we finally have the quasigeostrophic potential vorticity equation
\[ D_g q = \mathcal{X}, \]

where the potential vorticity is
\[ q = 2 \Omega \sin \varphi + \zeta + \frac{2 \Omega \sin \varphi}{\rho} \frac{\partial}{\partial z} \left( \frac{T}{S} \right) = 2 \Omega \sin \varphi + \Delta \psi, \]
where the nonconservative source of potential vorticity is
\[ \mathcal{X} = \frac{1}{a \cos \varphi} \left( \frac{\partial G}{\partial \lambda} \varphi - \frac{\partial}{\partial \varphi} (G \cos \varphi) \right) + \frac{2 \Omega \sin \varphi}{\rho} \frac{\partial}{\partial z} (\rho \frac{Q}{S}) . \]

6.6.4 Waves on a zonal flow

We now as usual consider the problem of small-amplitude linear waves on a zonal flow \( U(\varphi, z) [V = W = 0] \) with uniform static stability \( N^2 \) and a corresponding (modified) potential vorticity \( Q(\varphi, z) \) such that
\[ \frac{1}{a} \frac{\partial Q}{\partial \varphi} = \frac{2 \Omega}{a} \cos \varphi - \Delta U , \]
where we have again assumed that \( \varphi \) is slowly-varying compared to \( U \). (This problem was first formulated by Matsuno, JAS, 27, p871, 1970). The linearized potential vorticity equation is then
\[ \left( \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \lambda} \right) q' + \frac{\nu'}{a} \frac{\partial Q}{\partial \varphi} = \mathcal{X}' , \]
where \( \omega = U/(a \cos \varphi) \) is the angular velocity of the mean flow.

If we look for forced, stationary \((\partial/\partial t = 0)\) waves of zonal wavenumber \( s(\partial/\partial \lambda \equiv is)\); then
\[ is\omega \Delta \psi' + \frac{\partial Q/\partial \varphi}{a^2} \cos \varphi is \psi' = \mathcal{X}' . \]
If dissipation and internal forcing are zero so that \( \mathcal{X}' = 0 \), then this may be written as
\[ \mathcal{O}(\Psi) + \nu_s \psi = 0 , \]
where
\[ \psi' = \text{Re} \, \Psi(\varphi, z)e^{is\lambda}e^{z/2H} , \]
and where the operator \( \mathcal{O} \) is
\[ \mathcal{O} = \frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} \left( \cos \varphi \frac{\partial}{\partial \varphi} \right) + h^2 \sin^2 \varphi \frac{\partial^2}{\partial z^2} \]
where \( h = 2\Omega a/N \). \( \mathcal{O} \) has the character of a dimensionless Laplacian in \( \varphi \) and \( z \). Finally, \( \nu_s \), the “refractive index square” is
\[
\nu_s = \frac{1}{\omega} \frac{\partial Q}{\partial \phi} - \frac{h^2}{4H^2\sin^2 \varphi} \cdot \frac{s^2}{\cos^2 \varphi}.
\]

N.B. \(\nu_s\) is singular at the “critical line” where \(U = \omega = 0\). We can remove the singularity there by adding dissipation (nonzero \(\mathcal{X}'\)) to the problem. In fact (as we shall see) the processes of dissipation near to the critical line are complex and all we can hope to do with a linearized approach is to model them in a crude way. So we may as well do it as simply as possible; this is achieved by assuming Rayleigh friction and Newtonian cooling with equal coefficients \(\alpha\):

\[
(\mathcal{G}_\lambda', \mathcal{G}_\phi') = -\alpha(u', v')
\]

\[
Q' = -\alpha T',
\]

when we find easily that

\[
\mathcal{X}' = -\alpha \tilde{q}'.
\]

Then all we have to do is replace \(\omega\) by \(\omega - i\alpha/s\) in the wave equation.

If the basic state is slowly-varying in both \(\varphi\) and \(z\), then we will find WKB solutions of the form \(\Psi \approx e^{i\ell \varphi} e^{imz}\) where \(l\) and \(m\) are dimensionless wavenumbers and where

\[
\ell^2 + h^2 \sin^2 \varphi \cdot m^2 - \nu_s = 0.
\]

So we can have wave-like solutions in \(\varphi\) and \(z\) (real \(\ell\) and \(m\)) only if \(\nu_s > 0\). This is the equivalent of the Charney-Drazin criterion for spherical geometry (and, of course, has similar form). It requires \(0 < U < U_c\), where

\[
U_c = a \frac{\partial Q}{\partial \varphi} \left( \frac{h^2}{4H^2 \sin^2 \varphi} + \frac{s^2}{\cos^2 \varphi} \right)^{-1}.
\]

Fig 6.6.4 shows \(\nu_0\) calculated by Matsuno for the basic state shown. \(\nu_s\) becomes very large in the tropics where \(\omega\) becomes small. Karoly and Hoskins (J.Met.Soc.Japan, 60, p109, 1982) showed that wave rays (in the direction of the group velocity vector) tend to be refracted toward large \(\nu_s\); this is shown in the EP fluxes of Fig ?? for Matsuno’s profile and from Karoly and Hoskins’ calculation. Hoskins and Karoly showed that, even if \(\nu_s\) were positive at all heights, thus apparently allowing upward propagation everywhere, penetration to high levels is limited simply because most of the upward propagating
wave activity is refracted away from the vertical, mostly toward the equator (which, in fact, reflects the tendency of planetary waves to propagate along great circles, other things being equal).

“Refractive index diagnostics”—the practice of calculating \( \nu_s \) and interpreting the result in terms of wave propagation—is a useful tool in assimilating data. We shall see some of its applications later. One application seems to undermine what was said earlier about the penetration of planetary waves into the southern winter stratosphere. It was argued that this could be prevented when \( U \) becomes strong in midwinter. However, Fig 6.6.4 shows \( \nu \) calculated from southern hemisphere data by Randel (QJRMS, 1988); there is no sign of a major change in \( \nu \) through the winter. The reason is not really to do with spherical geometry, but with the shear contribution to \( \partial Q / \partial y = \beta - \partial^2 U / \partial y^2 \). If \( U \) is constant in \( y \), \( [\partial Q / \partial y] / U = \beta / U \to 0 \) as \( U \) becomes large, thus preventing propagation. However, if locally \( U \approx U_0 \cos \ell y \), \( \partial Q / \partial y \to \ell^2 \) (independent of \( U_0 \)) as \( U_0 \) becomes much larger than \( \beta / \ell^2 \), so that \( \nu \) loses its sensitivity to \( U \). However, there may be reasons to question the validity of the WKB approximation under such circumstances (since, in a strong narrow jet, \( \Psi \) may tend to take on the structure of the jet, thus violating the slowly-varying assumption).

6.6.5 EP flux in spherical geometry

From our linearized potential vorticity equation

\[
\left( \frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \lambda} \right) q' + \frac{\nu'}{a} \frac{\partial Q}{\partial \varphi} = \mathcal{X}',
\]

it is straightforward, using the same procedures as in the beta-plane case, to obtain the EP relation

\[
\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = \mathcal{D}
\]

where the density is

\[
A = \frac{1}{2} \rho a \cos \varphi \frac{q'^2}{\partial Q / \partial \varphi},
\]

the nonconservative term is

\[
\mathcal{D} = \rho a \cos \varphi \frac{q' \mathcal{X}'}{\partial Q / \partial \varphi},
\]
and the EP flux is
\[ F = (F_\varphi, F_z) = (-\rho u' \cos \varphi, \frac{2\Omega \rho}{S} \sin \varphi \cos \varphi v' T') . \]

The extra factor (as compared with the beta-plane result) of \( \cos \varphi \) can be understood as an indication of the fact that, in spherical geometry, the EP flux represents (aside from constant factors) a flux of angular, rather than linear, momentum.

6.6.6 Observations of stratospheric planetary waves and wave forcing of the mean flow

We have seen that stratospheric planetary waves are large in winter (early and late winter in the southern hemisphere), are dominated by the quasi-stationary waves and grow with height through the stratosphere. Figs 6.6.6 and 6.6.6 show the wave amplitudes from a global climatology for January (when the waves are large in the northern hemisphere) and September (maximum amplitudes in the southern hemisphere). The corresponding mean heat, momentum and EP fluxes and the EP flux divergences are shown in Figs 6.6.6 and 6.6.6, together with the separate contributions from the monthly-

mean and transient waves (these transient waves are the departures from the monthly-mean wave and include slow variations of the quasi-stationary modes as well as truly transient, travelling, disturbances).

The EP fluxes show the characteristic upward propagation/equatorward refraction we discussed from the theoretical models. The “transient” waves are the larger component in the troposphere and the equatorward refraction in the troposphere is clear (see the paper by Edmon, Hoskins and McIntyre, J.A.S., 1980, for a thorough discussion of the tropospheric observations). In the stratosphere, the stationary component is (marginally) the larger. Upward/equatorward EP flux means poleward heat flux / poleward momentum flux, as is evident from the figs. Qualitatively, the southern hemisphere waves
are similar in structure to the northern hemisphere waves (except as regards seasonal cycle, of course) but of consistently smaller amplitude.

Thus far, we have ignored the nature of the transients. Some of them are undoubtedly a simple manifestation of the fact that the tropospheric processes driving planetary waves (flow over topography, diabatic heating, nonlinear effects) are not in fact steady themselves. Other components (as we shall see) probably arise from nonlinear effects within the stratosphere. Nevertheless, the structure and frequencies of much of the transient wave field appears to indicate that the atmosphere is “ringing”; i.e., in normal modes. The atmosphere possesses (in theory) a whole sequence of normal mode frequencies; two of the large-scale ones of these are at 4 days (zonal wave 2) and 10 days (zonal wave 1). Fig 6.6.6 shows the observed structure of the component of the geopotential field at 1hPa at these frequencies and wavenumbers during periods in April/May 1981. The latitudinal structure is orderly, and in fact agrees very well with theoretical predictions, assuming they are normal modes. [See Salby, Rev. Geophys. Spa. Phys., 22, 209, 1984, for a review of travelling waves].

In January, the EP flux convergence in the middle and upper stratosphere corresponds to a force per unit mass of around \(2 \text{ ms}^{-1}/\text{day} \approx 2 \times 10^{-5} \text{ ms}^{-2}\). In steady state, this would drive a poleward residual flow \(\vec{v}_s \approx 0.2 \text{ ms}^{-1}\). This is of the same order as, but a factor of 2 smaller than, our crude estimate from Section 3. Given the uncertainties in our estimate of the required mean poleward flow required to explain the observed mean temperature structure, as well as the uncertainties in estimates of \(\nabla \cdot \mathbf{F}\) (implicitly, this involves calculation of \(q'\), since \(\nabla \cdot \mathbf{F} = \rho \vec{w} q'\), and we shall note below the problems in doing this accurately), all we can say from this is that the forcing of the mean stratospheric state by planetary waves is of the right magnitude (and sign) to explain the observed climatology; it may be enough to explain it entirely, but the data are not good enough for us to be sure.

We note here that the same cannot be true for the mesosphere. We saw that the heat budget required poleward flow of 5-10 ms\(^{-1}\) in the winter mesosphere and equatorward flow of around 5 ms\(^{-1}\) in summer. Since we have seen that stationary planetary waves are essentially absent in summer it seems clear that these cannot be providing the required forcing. (There are, at times, large amplitude travelling planetary waves in the summer...
mesosphere, but these are only present for short periods of time and cannot provide the required sustained forcing). In principle, planetary waves could provide the required forcing in the winter hemisphere but in fact the wave amplitudes decrease with height in the mesosphere (see Fig 6.6.6) and it seems unlikely that their effects would be large enough. We therefore need to seek other kinds of wave motion to explain the climatology of the mesosphere.

Another important issue to be addressed from Figs 6.6.6-6.6.6 concerns the decrease of EP flux with height. This is not evident from the EP flux diagrams (which are inconsistently scaled for clarity) but it is clear from Figs 6.6.6 and 6.6.6 that wave amplitudes do not grow anything like as $\rho^{-1/2}$, or from Figs 6.6.6 and 6.6.6 that $\overline{\varepsilon' \theta'}$ does not grow as $\rho^{-1}$, as theory says an undissipated wave should. It is clear, therefore, that the waves are being dissipated rather strongly. We have included radiative dissipation in our linear calculations. There is no doubt that this makes a contribution to the real damping of the waves; it is doubtful, however, whether it is strong enough to explain the observed decay with height of wave activity (simple linear models usually give amplitudes too large in the upper stratosphere). There are in fact other dissipative processes acting; consideration of these brings us to the thorny question—which we have glossed over thus far—of the behavior near the “critical line” where $\bar{u} = 0$. 

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