2 Transport of heat, momentum and potential vorticity

2.1 Conventional mean momentum equation

We'll write the inviscid equation of zonal motion (we'll here be using log-pressure coordinates with \( z = -H \ln p/p_0 \), with \( H \) constant) as

\[
\frac{du}{dt} - f v = -\frac{\partial \phi}{\partial x}
\]  

(1)

and the continuity equation

\[
\nabla \cdot \rho u = 0
\]  

(2)

where \( \rho(z) = p/gH \). We introduce the “zonal mean” of any variable \( a \) as

\[
\bar{a}(y, z, t) = \frac{1}{L} \int_0^L a(x, y, z, t) \, dx ,
\]  

(3)

(where we assume periodic behavior in \( x \) such that \( a(L, y, z, t) = a(0, y, z, t) \) — note that this means that \( \frac{\partial a}{\partial x} = 0 \) and then the eddy component is

\[
a'(x, y, z, t) = a(x, y, z, t) - \bar{a}(y, z, t) .
\]

It follows that, by definition,

\[
\overline{a'} = 0 .
\]  

(4)

Note that the definition of the “eddy” variable is contingent on the definition of “mean.” Then the mean of (1), invoking (2), becomes

\[
\frac{\partial \bar{u}}{\partial t} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} - f \bar{v} = -\frac{1}{\rho} \nabla \cdot (\rho \overline{u'u'}) ,
\]  

(5)

thus introducing the concept of “eddy momentum flux” \( \rho \overline{u'u'} = (\rho u'u', \rho u'w') \) (it has no zonal component in our zonal mean formulation). The way in which the eddies influence the mean state is evidently equivalent to a body force \(-\nabla \cdot (\rho \overline{u'u'})\) acting on the mean zonal motion. This flux term clearly represents the advection, by the eddy component of the meridional velocity, of the eddy component of zonal momentum. The zonal momentum per unit mass is \( u \), so the advective flux of momentum across a coordinate surface (of constant \( y \) and \( z \)) is \( \rho uu \), whence its zonal mean is \( \overline{\rho uu} = \rho \bar{u}u + \rho \overline{u'u'} \); hence it is natural to regard \( \rho \overline{u'u'} \) as the eddy flux of momentum. It is important to realise, however, that this definition of eddy momentum flux is a consequence of our definition (3) of what is meant by “mean,” specifically, that the average is taken at \textit{at constant pressure} (constant \( z \) in log-p coordinates).
One could easily define other means: since transport is fundamentally a Lagrangian process, one might try taking some kind of Lagrangian mean, but we will not do that here\(^1\). One can take a different integration path in (3) — the way we have done it is to integrate at constant latitude and height (\textit{i.e.}, constant pressure) — by using a different coordinate system, in which case the whole problem can look rather different, as we are about to see.

### 2.2 Wave, mean flow interaction in isentropic coordinates

In \((x, y, \theta)\) coordinates, the inviscid zonal equation of motion is\(^2\)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \theta \frac{\partial u}{\partial \theta} - f v = -\frac{\partial M}{\partial x}
\]

and that of continuity

\[
\frac{\partial \sigma}{\partial t} + \frac{\partial}{\partial x} (\sigma u) + \frac{\partial}{\partial y} (\sigma v) + \frac{\partial}{\partial \theta} (\sigma \theta) = 0
\]

where \(\sigma = -g^{-1} \partial p / \partial \theta\) is the isentropic coordinate density, and

\[
M = c_p T + g z = \Pi \theta + \phi
\]

is the Montgomery potential, with \(\Pi(p) = c_p \left( p/p_0 \right)\) the Exner function and \(\phi = g z\). (Note that \(z\) is “real” height here, not log-pressure height.) It follows from (8), with some juggling and the assumption of hydrostatic balance, that

\[
\frac{\partial M}{\partial \theta} = \Pi.
\]

Now our definition of zonal mean is

\[
\bar{a}(y, \theta, t) = \frac{1}{L} \int_0^L a(x, y, \theta, t) \, dx
\]

where the integral is now, of course, taken at constant \(\theta\), and the eddy term is

\[
a'(x, y, \theta, t) = a(x, y, \theta, t) - \bar{a}(y, \theta, t).
\]

\(^1\)If you are interested in the Lagrangian perspective, there are some additional notes on this on the class web page. The conceptually powerful “generalized Lagrangian mean” theory of wave, mean flow, interaction can be found in Andrews & McIntyre, \textit{J. Fluid Mech.}, 1978 (though it is not for the faint hearted!).

\(^2\)The derivation of the \(\theta\)-coordinate equations is discussed in Appendix 3A in Andrews, Holton and Leovy [1987] (at the end of Chapter 3).
It makes sense to define mean meridional velocities in a density-weighted sense:

\[ \bar{v}^* = \bar{\sigma}^{-1} (\sigma v^*) ; \quad \bar{\theta}^* = \bar{\sigma} \left( \frac{\sigma \theta^*}{\bar{\sigma}} \right) \]  

(10)

since then the mean continuity equation has no eddy flux terms:

\[ \frac{\partial \bar{\sigma}}{\partial t} + \frac{\partial}{\partial y} (\bar{\sigma} \bar{v}^*) + \frac{\partial}{\partial \theta} \left( \bar{\sigma} \bar{\theta}^* \bar{u} \right) = 0 . \]

The momentum equation (6), on multiplication by \( \sigma \), use of (7), and zonal averaging, gives

\[ \frac{\partial}{\partial t} (\bar{\sigma} \bar{u}) + \frac{\partial}{\partial y} (\bar{\sigma} \bar{v}^* \bar{u}) + \frac{\partial}{\partial \theta} \left( \bar{\sigma} \bar{\theta}^* \bar{u} \right) - f \bar{\sigma} \bar{v}^* = \frac{\partial}{\partial y} \left( (\sigma v)^' \bar{u} \right) - \frac{\partial}{\partial \theta} \left( (\sigma \theta)^' \bar{u} \right) - \sigma \frac{\partial M}{\partial x} . \]  

(11)

The first two terms on the rhs are easily identifiable as advection by the eddy mass fluxes, but what about the third term? Since \( \partial M/\partial x = 0 \), it is an eddy term

\[ \sigma \frac{\partial M}{\partial x} = \sigma \frac{\partial M'}{\partial x} \]

and, in fact, the term can be written, using the definition of \( \sigma \) and (8)

\[ -\sigma \frac{\partial M}{\partial x} = \frac{1}{g} \frac{\partial p}{\partial \theta} \frac{\partial M}{\partial x} \]

\[ = \frac{\partial}{\partial \theta} \left( \frac{1}{g} \frac{\partial M}{\partial x} \right) - \frac{1}{g} \frac{\partial^2 M}{\partial x \partial \theta} . \]

But \( \partial M/\partial \theta = \Pi(p) \), so

\[ \frac{\partial^2 M}{\partial x \partial \theta} = \frac{\partial}{\partial x} \frac{\partial \Pi(p)}{\partial p} = \frac{\partial}{\partial x} \frac{dp}{d\Pi} \frac{\partial p}{\partial x} \]

\[ = \frac{\partial}{\partial x} \int p \frac{dp}{d\Pi} dp = 0 , \]

hence

\[ -\sigma \frac{\partial M}{\partial x} = \frac{\partial}{\partial \theta} \left( \frac{1}{g} \frac{\partial M}{\partial x} \right) = \frac{\partial}{\partial \theta} \left( \frac{1}{g} \frac{\partial M'}{\partial x} \right) . \]

In total, therefore, (11) becomes

\[ \frac{\partial}{\partial t} (\bar{\sigma} \bar{u}) + \frac{\partial}{\partial y} (\bar{\sigma} \bar{v}^* \bar{u}) + \frac{\partial}{\partial \theta} \left( \bar{\sigma} \bar{\theta}^* \bar{u} \right) - f \bar{\sigma} \bar{v}^* = \frac{\partial}{\partial y} \left( (\sigma v)^' \bar{u} \right) - \frac{\partial}{\partial \theta} \left( (\sigma \theta)^' \bar{u} \right) - \sigma \frac{\partial M'}{\partial x} ; \]  

(12)
there is an eddy momentum flux whose $y$ and $\theta$ components are given by
\[
\left[ (\sigma v)u', (\sigma \theta)u' - \frac{1}{g} p' \frac{\partial M'}{\partial x} \right].
\] (13)

The additional, non-advective, term in (13) represents the form stress acting on the isentropic surface. If one has, e.g., surface topography $z(x, y)$ there is an inviscid stress on the earth’s surface given by $\rho \sin \alpha = p \frac{\partial z}{\partial x}$, where $\alpha$ is the angle of the local surface with respect to the horizontal; accordingly there is a form drag of $-p \frac{\partial z}{\partial x}$ acting on the atmosphere above. Now,
\[
\frac{p' \frac{\partial M'}{\partial x}}{p} = \frac{\partial M}{\partial x} = -p \left( \frac{\partial \Pi}{\partial p} \frac{\partial p}{\partial \theta} + g \frac{\partial z}{\partial x} \right)
\]
\[
= \theta p \frac{\partial \Pi}{\partial \theta} \frac{\partial z}{\partial \theta} + g \frac{\partial z}{\partial x} = g \frac{\partial z}{\partial x},
\]
since we just established that the first term is zero. Hence
\[
-\frac{1}{g} p' \frac{\partial M'}{\partial x} = -p \frac{\partial z}{\partial x}.
\]
Since $\frac{\partial z}{\partial x}$ here is just the slope of the isentropic surface, we can therefore immediately identify this extra term as the form drag acting on an isentropic surface due to the presence of the eddies. So, in a different coordinate system, eddy transport of momentum can look quite different, physically as well as mathematically.

2.3 QG momentum, heat, and PV transport

2.3.1 The quasigeostrophic equations on a $\beta$-plane

We consider motion with characteristic horizontal length scale $L$, height scale $h$, velocity scale $U$, time scale $\geq L/U$ on a $\beta$-plane for which the Coriolis parameter is $f = f_0 + \beta y$. We make the assumptions that:

(i) the Rossby number $Ro = U/f_0 L$ is small,

(ii) $\beta L/f_0 \leq Ro$ (which, in a spherical context, means that $L << a$, where $a$ is the Earth’s radius),

(iii) the isentropic slopes $|\theta_x|/|\theta_z|$ and $|\theta_y|/|\theta_z|$ are $\leq Ro (h/L)$, (otherwise vertical motions would not be small), and

\footnote{Under the “shallow atmosphere” approximation which all these equations assume, the slope is small, so $\frac{\partial z}{\partial x} = \tan \alpha \approx \sin \alpha$.}
(iv) the static stability $\theta_z$ is, to leading order, a function of $z$ only [this stems from (iii)], so we write $\theta = \Theta_0(z) + \hat{\theta}$, where $\hat{\theta}$ is $O(Ro)$ smaller than $\Theta_0$. We then define a background geopotential, in hydrostatic balance with the background $\Theta_0$:

$$\Phi_z = \frac{g}{c_p T_s} \Pi \Theta_0 = \frac{\kappa}{H} \Pi \Theta_0 .$$

Under these assumptions, we find that the leading order equations give geostrophic balance, in which we write the *geostrophic* velocities [*i.e.*, the leading order terms in a Rossby number expansion of $(u, v, w)$] as

$$u = -\psi_y ; v = \psi_x ; w = 0 ;$$

where $\psi$ is the geostrophic streamfunction, which we define to be

$$\psi = [\phi - \Phi(z)] / f_0 .$$

Hydrostatic balance then gives us

$$\psi_z = \frac{\kappa \Pi}{f_0 H} \theta - \frac{1}{f_0} \Phi_z = \frac{\kappa \Pi}{f_0 H} [\theta - \Theta_0(z)] = \frac{\kappa \Pi}{f_0 H} \hat{\theta} .$$

At next order, we obtain our quasigeostrophic equations. The equations of motion are

$$D_g u - \beta y v - f_0 v_a = G(x) ,$$

$$D_g v + \beta y u + f_0 u_a = G(y) ,$$

where $G$ represents frictional or other forces per unit mass, $D_g$ is the time derivative following the *geostrophic* flow

$$D_g \equiv \partial_t + u \partial_x + v \partial_y$$

and $(u_a, v_a, w_a)$ is the *ageostrophic* velocity (*i.e.*, the difference between the actual velocity and the geostrophic one). Similarly, the thermodynamic equation becomes

$$D_g \theta + w_a \Theta_{0,z} = c_p \Pi^{-1} Q = (\rho \Pi)^{-1} \mathcal{J} ,$$

where $\mathcal{J}$ is the diabatic heating rate per unit volume (and $Q = \mathcal{J} / \rho c_p$ is the heating rate expressed in units of K per unit time). From these, we can readily derive the equation for quasigeostrophic potential vorticity, $q$:

$$D_g q = \mathcal{X} ,$$

where

$$q = f_0 + \beta y + v_x - u_y + \frac{f_0}{\rho} \left( \rho \frac{\hat{\theta}}{\Theta_{0,z}} \right)_z = f_0 + \beta y + \Delta^2 \psi .$$
and

$$\mathcal{X} = G_x^{(y)} - G_y^{(x)} + \frac{f_0}{\rho} \left( \Pi^{-1} \frac{\mathcal{J}}{\Theta_{0,z}} \right)_z,$$  \hspace{1cm} (21)$$

where

$$\Delta^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial}{\partial z} \right),$$  \hspace{1cm} (22)$$

with

$$N^2 = \frac{\kappa}{H} \Pi \Theta_{0,z}$$

being the square of the buoyancy frequency. Eq. (19) tells us that, for conservative flow ($\mathcal{G} = 0$, $\mathcal{J} = 0$, whence $\mathcal{X} = 0$), $q$ is conserved following the geostrophic flow. When the flow is not conservative, $\mathcal{X}$ represents the local sources or sinks of $q$ arising from viscous and/or diabatic effects.

### 2.3.2 PV fluxes and the Eliassen-Palm theorem

Consider perturbations to a steady, zonally-uniform basic state

$$U = U(y, z) ; \ \Theta = \Theta(y, z) ; \ \Psi = \Psi(y, z)$$

where

$$\Psi_y = -U ; -\frac{\kappa \Pi}{H} \Theta_y = f_0 U_z.$$  \hspace{1cm} (23)$$

The basic state PV is:

$$Q(y, z) = f_0 + \beta y + \Psi_{yy} + \frac{1}{\rho} \left( \rho \frac{f_0^2}{N^2} \Psi_z \right)_z = f_0 + \beta y + \Delta^2 \Psi.$$  \hspace{1cm} (24)$$

Write

$$\psi = \Psi + \psi'(x, y, z, t)$$

where $\psi'$ is a small perturbation. Now, $v' = \psi'_x$ and

$$q' = \Delta^2 \psi' = \psi''_{xx} + \psi''_{yy} + \frac{1}{\rho} \left( \rho \frac{f_0^2}{N^2} \psi'_z \right)_z,$$  \hspace{1cm} (25)$$

therefore

$$\overline{v'q'} = \overline{\psi''_{xx} \psi''_{xx}} + \overline{\psi''_{xy} \psi''_{yy}} + \frac{1}{\rho} \overline{\psi''_x \left( \rho \frac{f_0^2}{N^2} \psi'_z \right)_z}.$$  \hspace{1cm} (26)$$

Simple manipulation yields the following identities:
1.  
\[ \psi_x' \psi_x' = \frac{1}{2} \left[ (\psi_x')^2 \right]_x = 0 ; \]

2.  
\[ \psi_x' \psi_y' = \left( \psi_x' \psi_y' \right)_y - \psi_y' \psi_x' = \left( \psi_x' \psi_y' \right)_y - \frac{1}{2} \left[ (\psi_y')^2 \right]_x \]
\[ = \left( \psi_x' \psi_y' \right)_y ; \]

and

\[ \psi_x' \frac{1}{\rho} \left( \frac{\rho q^2}{N^2} \psi_z' \right)_z = \frac{1}{\rho} \left( \frac{\rho f_0^2}{N^2} \psi_x' \psi_z' \right)_z \]
\[ = \frac{1}{\rho} \left( \frac{\rho f_0^2}{N^2} \psi_x' \psi_z' \right)_z - \frac{f_0^2}{N^2} \psi_z' \psi_x' \]
\[ = \frac{1}{\rho} \left( \frac{\rho f_0^2}{N^2} \psi_x' \psi_z' \right)_z - \frac{f_0^2}{2N^2} \left[ (\psi_z')^2 \right]_x \]
\[ = \frac{1}{\rho} \left( \frac{\rho f_0^2}{N^2} \psi_x' \psi_z' \right)_z . \]

Therefore, from (25),
\[ \rho \overline{v' q'} = \nabla \cdot \mathbf{F} . \quad (26) \]

where
\[ \mathbf{F} = \left( \begin{array}{c} F^{(u)} \\ F^{(z)} \end{array} \right) = \left( \begin{array}{c} \rho \psi_x' \psi_y' \\ \rho \psi_y' \psi_z' / N^2 \end{array} \right) = \left( \begin{array}{c} -\rho \overline{v' q'} \\ \rho f_0 \overline{v' q'} / \overline{\psi_0 q} \end{array} \right) . \quad (27) \]
\[ \mathbf{F} \] is known as the ELIASSEN-PALM flux. Note that the northward component of \( \mathbf{F} \) is (minus) the northward eddy flux of zonal momentum, while the vertical component is proportional to the northward eddy flux of heat, \( \overline{v' T} \).

Now, assume that the perturbations are small, such that the QGPV equation (19) can be linearized to give
\[ (\partial_t + U \partial_x) q' + v' Q_y = \mathcal{X}^t ; \]

multiply by \( q' \) and average:
\[ \frac{1}{2} \left( \overline{q'^2} \right)_t + \overline{v' q'} Q_y = \overline{\mathcal{X}^t} . \quad (28) \]

If we define
\[ A = \rho \frac{1}{2} \overline{q'^2} / Q_y \text{ and } D = \rho \overline{v' \mathcal{X}^t} / Q_y, \]
then, using (26),

\[ A_t + \nabla \cdot F = D. \]  \hspace{1cm} (29)

Eq. (29) is the ELIASSEN-PALM RELATION. It is a conservation law for zonally-averaged wave activity whose density is \( A \). Note that \( D \rightarrow 0 \) for conservative flow.

The significance of this relation is that it gives us a measure of the flux of wave activity through wave propagation. For example, if the waves are conservative (\( D = 0 \)) then \( A \) must increase with time wherever \( F \) is convergent and decrease wherever it is divergent. Thus \( F \) is a meaningful measure of the propagation of wave activity from one place to another. This becomes most obvious for almost-plane waves (WKB theory) when, as we shall see,

\[ F = c_g A \]

where \( c_g \) is group velocity, in which case

\[ A_t + \nabla \cdot c_g A = D. \]

However, note that \( F \) remains valid as a measure of the flux of wave activity even when WKB theory is not valid and we cannot even define group velocity.

[N.B. there are some subtleties to these arguments if \( Q_y \) changes sign anywhere, since \( A \) is then not positive definite (and so increasing \( A \) does not necessarily mean increasing wave amplitude). Indeed, if this occurs, the basic state may be unstable—the Charney-Stern necessary condition for instability can in fact be readily obtained from (29).]

### 2.3.3 The Eliassen-Palm theorem

For waves which are steady (\( A_t = 0 \)), of small amplitude and conservative (\( D = 0 \)), the flux \( F \) is nondivergent. This is, through (26), the same thing as saying that the northward flux of quasigeostrophic potential vorticity vanishes under these conditions [a result we could of course have obtained from (28) without involving \( F \)].

### 2.3.4 Potential vorticity transport and the nonacceleration theorem

We now consider the problem of how eddies (in this case, quasigeostrophic eddies) impact the zonal mean circulation. We return to eddies which may not be small and to the QGPV budget, which becomes, on zonal averaging

\[ \bar{q}_t + (\bar{v} \overline{q'})_y = \bar{X}. \]  \hspace{1cm} (30)

Note that, unlike (5), there is no mean advection term in (30). This is because there is no advection by \( w \) in this quasigeostrophic case and \( \bar{v} = \overline{\psi_x} = 0 \). Similarly, the eddy flux contains no vertical component, because of QG scalings.
The dynamical influence of the eddies on mean potential vorticity, therefore, is described entirely\(^4\) by the northward flux \(\bar{v}q'\). Given the equivalence between \(\bar{pv}'q'\) and \(\nabla \cdot \mathbf{F}\), we could make the same statement about \(\nabla \cdot \mathbf{F}\)—which informs us immediately that \(\mathbf{F}\) is telling us something about wave transport as well as propagation.

Now, we know from the Eliassen-Palm theorem that if the waves are everywhere

(I) of steady amplitude,

(II) conservative, and

(III) of small amplitude, (so that terms of \(O(\epsilon^3)\) in the wave activity budget are negligible)

then \(\mathbf{F}\) is nondivergent and \(\bar{v}q' = 0\). Under these conditions, therefore, the budget equation for zonally-averaged QGPV is independent of the waves (if we assume that \(\bar{X}\) is so independent). Now,

\[
\bar{q} = f + \Delta^2(\bar{\psi})
\]

therefore

\[
\Delta^2(\bar{\psi}_t) = \bar{q}_t = \bar{X}.
\]

If \(\bar{q}_t\) is independent of the waves, then so is \(\Delta^2(\bar{\psi}_t)\); since \(\Delta^2\) is an elliptic operator, the solution of the above equation for \(\bar{\psi}_t\) invokes boundary conditions. If, however, we invoke the further condition that

(IV) the boundary conditions on \(\bar{\psi}_t\) are independent of the waves

then \(\bar{\psi}_t\) is everywhere independent of the waves. Since \(\bar{u} = -\bar{\psi}_y\) and \(\bar{\theta} = (f_0 T_s / g \Pi) \bar{\psi}_z\), it then follows that, under conditions (I)-(IV), the tendencies of the mean geostrophic flow, mean temperature and mean QGPV are independent of the waves. This is known as the nonacceleration theorem and conditions (I)-(IV) are sometimes known as “nonacceleration conditions”.

### 2.3.5 Zonal mean momentum and heat budgets: Conventional Eulerian-mean approach

In a similar fashion, we can take the zonal mean of the quasigeostrophic momentum and heat equations. The zonal mean of the first of (17) gives us

\[
\bar{u}_t - f_0 \bar{v}_a = \mathbf{G}^{(x)} - (u'v')_y,
\]  

(31)

\(^4\)One should add the caveat that, in principle, the eddies could also influence \(\bar{X}\) (e.g., eddies could modify mean precipitation).
where we have used $\bar{v} = \bar{\psi}_x = 0$ (the mean ageostrophic wind is not zero, however). Similarly, the heat equation (18) gives

$$\frac{\partial \bar{\psi}}{\partial t} + \bar{\psi}_a \Theta_{0z} = (\rho \Pi)^{-1} \mathcal{J} - (\bar{\psi} \bar{\theta}')_y .$$

(32)

The mean flow equations are closed by the continuity equation

$$\bar{v}_{a,y} + \frac{1}{\rho} (\rho \bar{\psi}_a)_z = 0$$

(33)

and the thermal wind equation

$$f_0 \bar{u}_z = -\frac{\kappa \Pi}{H} \bar{\theta}_y .$$

(34)

(The two latter equations are linear and so the zonal averaging is trivial).

The evolution of the zonal mean state in the presence of eddies is therefore specified by (31) - (34). In this case, the effects of wave transport are manifested in two terms — the convergence of the eddy fluxes of momentum $\bar{u}' \bar{v}'$ and heat $\bar{v}' \bar{\theta}'$ (strictly, the eddy flux of heat is $\rho c_p \bar{v}' \bar{T}'$, but we shall refer to $\bar{v}' \bar{\theta}'$ in this way for simplicity). Both these terms force the mean flow equations and it is important to note that the whole system is coupled, i.e., the heat fluxes can impact on the mean winds just as much as can the momentum fluxes. Thermal wind balance requires this to be true. Consider, for example, an upward propagating wave with $\bar{v}' \bar{\theta}' \neq 0$ but $\bar{u}' \bar{v}' = 0$. The mean state could not respond with a changing mean temperature only; thermal wind balance demands a corresponding change in $\bar{u}$. From (31) this could only be achieved through an ageostrophic meridional circulation, which would impact on both the momentum and heat budgets. Thus, the waves will not only drive $\bar{u}_t$ and $\bar{\theta}_t$, but also $\bar{v}_a$ and $\bar{w}_a$ (except in the unlikely case where the eddy forcing terms conspire not to disturb thermal wind balance).

To put the same statements into mathematics, what (31) - (34) give us is a set of 4 equations in the 4 unknowns $\bar{u}_t$, $\bar{\theta}_t$, $\bar{v}_a$ and $\bar{w}_a$ in terms of the two eddy driving terms. In general, when both the eddy flux terms are nonzero, there is no simple way of saying which forcing achieves what response.

Moreover, the central role of the potential vorticity flux—obvious in the PV budget—is not at all obvious here. Indeed, we have seen from the mean potential vorticity budget that under “nonacceleration conditions” $\bar{u}_t$ and $\bar{\theta}_t$, must be zero under these conditions. What must (and does) happen under such circumstances is that the eddies induce ageostrophic mean motions whose effects in (31) and (32) exactly balance the eddy flux terms. All in all, this approach to the mean momentum and heat budgets is not giving us much insight into what is going on.
2.4 Transformed Eulerian-mean theory

The most important lesson to learn from the isentropic-coordinate analysis we did earlier is that the definition of “mean,” and consequently of concepts like eddy fluxes, is non-unique. We can take this message to heart by allowing ourselves some flexibility in defining what we mean by the mean meridional circulation.

We begin by noting that, from (33), we may define an ageostrophic mean streamfunction \( \chi_a \) such that

\[
(\bar{v}_a, \bar{w}_a) = \left[ \frac{1}{\rho} \frac{\partial (\rho \chi_a)}{\partial z}, -\frac{\partial \chi_a}{\partial y} \right].
\]  

(35)

Now, we look for a more revealing way of defining “mean circulation”. To make things as simple as possible, we insist that our modified circulation be nondivergent also, so we write

\[
(\bar{v}_*, \bar{w}_*) = \left[ \frac{1}{\rho} \frac{\partial (\rho \chi_*)}{\partial z}, -\frac{\partial \chi_*}{\partial y} \right],
\]  

(36)

where the new streamfunction is

\[
\chi_* = \chi_a + \chi_c.
\]

If we substitute this into (32), we obtain

\[
\frac{\partial \theta}{\partial t} + \bar{w}_* \Theta_{0z} = (\rho \Pi)^{-1} \mathcal{J} - \frac{\partial}{\partial y} (v' \theta') - \Theta_{0z} \frac{\partial \chi_c}{\partial y}.
\]

Noting that \( \Theta_0 = \Theta_0(z) \), it follows that if we make the choice

\[
\chi_c = \frac{v' \theta'}{\Theta_{0z}},
\]

so that \( \chi_* \)—the streamfunction of the so-called residual circulation—is

\[
\chi_* = \chi_a - \frac{v' \theta'}{\Theta_{0z}},
\]  

(37)

we obtain the thermodynamic equation

\[
\frac{\partial \theta}{\partial t} + \bar{w}_* \Theta_{0z} = (\rho \Pi)^{-1} \mathcal{J}.
\]  

(38)

We have thus succeeded in deriving a mean heat equation in which there are no explicit eddy terms; heat is transported solely through the mean vertical “residual” motion. It might be thought, of course, that the eddy terms are still there, implicit in \( \bar{w}_* \); but this was also true of \( \bar{w}_a \), which, as noted earlier, is in general influenced by the eddies. What we have done is to redefine this influence so as to put the mean heat budget into its simplest possible form.
We now need to complete our transformed system of equations. The continuity equation is
\[ \frac{\partial \bar{v}_s}{\partial y} + \frac{1}{\rho} \frac{\partial (\rho \bar{w}_s)}{\partial z} = 0 \] (39)
[we arranged this by (36)]. The thermal wind equation stays as before, i.e. :
\[ f_0 \bar{u}_z = -\frac{\kappa \Pi \bar{v}_s}{H} \] (40)

The momentum equation is less trivial. We need to replace \( \bar{v}_a \) using (36); the result is
\[ \frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_s = \mathcal{G}_x + \frac{1}{\rho} \nabla \cdot \mathbf{F}, \] (41)
where \( \mathbf{F} \) is the Eliassen-Palm flux.

This transformation—which is nothing more than a different way of writing the same equations—makes the role of the eddies look rather different. We now have, in (38) - (41), a set of equations for \( \bar{v}_s, \bar{w}_s, \partial \bar{u}/\partial t \) and \( \partial \bar{v}/\partial t \) in which the only term representing eddy forcing is a term \( \rho^{-1} \nabla \cdot \mathbf{F} \), which appears as an effective body force (per unit mass) in the mean momentum equation. [We could, for example, redefine \( \mathcal{G}_x \) to absorb this term]. It is clear, therefore, that under nonacceleration conditions when \( \nabla \cdot \mathbf{F} = 0 \) and the boundary conditions are independent of wave-dependent terms, \( \bar{v}_s, \bar{w}_s, \partial \bar{u}/\partial t \) and \( \partial \bar{v}/\partial t \) are independent of the waves.

When nonacceleration conditions are not satisfied, the transformed equations offer a more transparent approach to the problem simply because the single term represented by the effective body force \( \rho^{-1} \nabla \cdot \mathbf{F} \) entirely summarizes the eddy forcing of the mean state (there being no thermal eddy forcing to confuse the issue). In fact, this formulation gives us another interpretation of \( \mathbf{F} \): as an eddy flux of (negative, i.e., easterly) momentum which, because of the properties we have just described, is a more reliable measure of the wave transport of momentum than \( \bar{u}' \bar{v}' \) alone. Moreover, we now see that the Eliassen-Palm flux gives us a unified picture of wave propagation (through its role in wave activity conservation) and transport (through its interpretation as a momentum flux). This perspective is conceptually very powerful, as we shall see. Moreover, through the relationship (26) we immediately recover the central role of PV fluxes in the eddy forcing of the mean state.

**2.4.1 Relationship to isentropic mean perspective**

Finally, note again that, while the perspective which regards \( \mathbf{F} \) as a momentum flux may seem to be the result of mere mathematical juggling, it is rather more than that. It might seem odd to have what looks like an eddy heat flux appearing in the vertical momentum flux in this formulation. But it can be shown (see relevant Problem Set) that, for small-amplitude eddies under adiabatic conditions:

12
1. this heat flux term corresponds to the form stress term that appeared in the isentropic coordinate formulation, and

2. that the “residual circulation” corresponds to the density-weighted mean circulation isentropic coordinates. [It is also, under these assumptions, directly related to the Lagrangian mean motion.]

Thus, aside from the mathematical advantages of the TEM set of equations over the “conventional” set, the formulation is also describing the physics rather well. [You might well ask: why not use the isentropic coordinate version? The answer is that, for wave problems, isentropic coordinates are a bit unwieldy.]

### 2.4.2 Boundary conditions on the TEM equations

Finally, a short but important note about boundary conditions on the TEM flow. At a flat lower boundary, the boundary condition \( w = 0 \) translates straightforwardly, under conventional averaging, to \( \overline{w} = 0 \). For the TEM equations, however, (36) and (37) tells us that

\[
\overline{w} = \frac{\partial}{\partial y} \left( \frac{v' \theta'}{\Theta_0} \right)
\]

which is nonzero whenever there is a heat flux along the boundary. So, in this respect, the TEM framework is the more complicated (though it is actually telling us something about the dynamics near the boundary — see Problem Set).

As we shall see, on a lower boundary with topography, the opposite may be true. In general \( \overline{w} \) is nonzero on such a boundary.