5 Instability of zonal flows (QG)

(Holton, Ch. 8; a detailed exposition of the theory is given in Pedlosky, Ch 7.7.)

5.1 Violating the stability constraint: Barotropic and baroclinic instability

As we have seen zonal flows are stable to, inviscid, adiabatic, normal mode, QG disturbances if the PV gradient is single signed and the upper and lower boundaries are isentropic. Since the QGPV is

\[ q = f + \frac{\partial^2 \psi}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \]

and for a zonal flow \( U = -\partial \psi / \partial y \), the PV gradient is

\[ \frac{\partial q}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \right) \]  \( \text{(1)} \)

For a barotropic flow (no \( T \) gradients, and \( \partial U / \partial z = 0 \)), non-isentropic boundaries are not an issue, and the PV gradient is

\[ \left( \frac{\partial q}{\partial y} \right)_{\text{barotropic}} = \beta - \frac{\partial^2 U}{\partial y^2} \]  \( \text{(2)} \)

In the case \( \beta = 0 \), such a zonal flow is necessarily stable unless the curvature term has both signs within the fluid. If the curvature is everywhere finite, this yields the inflection point theorem: the flow is stable unless \( U(y) \) has an inflection point where \( \partial^2 U / \partial y^2 = 0 \). Introduction of \( \beta > 0 \) is a stabilizing influence: barotropic instability is then possible only if the curvature term is of the correct sign and of sufficient magnitude to overcome \( \beta \) in (2), somewhere in the flow.

In the absence of barotropic curvature,

\[ \frac{\partial q}{\partial y} = \beta - \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial U}{\partial z} \right) \]  \( \text{(3)} \)

and baroclinic instability is possible if the upper and lower boundaries are not isentropic, or if the vertical “curvature” (modified by \( \rho f_0^2 / N^2 \)) is sufficient
to overcome $\beta$ in (3). In the extratropical troposphere, the curvature is not usually sufficient to do so in the free atmosphere, but the stability condition is violated by the presence of temperature gradients on the lower boundary.

In the presence of both barotropic and baroclinic curvature, both curvature terms in (1) may contribute to changing the sign of $\partial q/\partial y$, resulting in mixed barotropic-baroclinic instability.

5.2 Baroclinic instability: The Eady problem

The simplest example of baroclinic instability (and one which is actually more relevant to the real atmosphere than it might appear) in a continuously stratified fluid is the Eady problem. The mean state has the following characteristics:

1. It is Boussinesq ($\rho = \text{constant in inertial terms}$)
2. Inviscid, adiabatic flow on an $f-$ plane ($f = f_0$ is constant: $\beta = 0$)
3. Uniform buoyancy frequency: $N^2 = -g \rho_0 (\partial \rho_0 / \partial z)$ is constant
4. Rigid horizontal upper and lower boundaries at $z = \pm \frac{1}{2} D$, on which $w = 0$.
5. Basic state comprises a zonal flow that increases linearly with height: $u_0 = \Lambda z$
6. Basic state density in thermal wind balance with the wind:

$$\rho_0 = \rho_{00} \left[ 1 + \frac{1}{g} \left( f \Lambda y - N^2 z \right) \right]$$

where $\rho_{00}$ is constant, so $\partial \rho_0 / \partial y = f \Lambda \rho_0 / g$: $\rho_0$ increases uniformly with latitude everywhere.

The basic state geostrophic streamfunction is

$$\Psi = - \int U(z) \, dy = -\Lambda z y + \text{constant}$$

and the basic state QGPV is

$$Q = f_0 + \frac{\partial^2 \Psi}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \Psi}{\partial z^2} = f_0$$
so the basic state has no PV gradient—this is the defining characteristic of the Eady problem. It then follows from the stability criterion that this flow must be stable unless there are density gradients on the boundaries, which of course there are.

The perturbation QGPV equation is

\[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) q' + v' \frac{\partial Q}{\partial y} = 0. \]

But since \( \partial Q/\partial y = 0 \), if \( q' = 0 \) everywhere at some initial time, then

\[ q' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{f_0^2}{N^2} \frac{\partial^2 \psi'}{\partial z^2} = 0. \]

If we look for separable modal solutions, wave-like in the horizontal, of the form

\[ \psi' = \text{Re} [\Phi(z)e^{i(kx+ly-\kappa t)}] \] (4)

then

\[ \frac{d^2 \Phi}{dz^2} - \frac{N^2}{f_0^2} \kappa^2 \Phi = 0 \]

where \( \kappa = \sqrt{k^2 + l^2} \). Then \( \Phi \sim \exp(\pm N \kappa z/f_0) \), or

\[ \Phi(z) = A \cosh \left( \frac{N \kappa}{f_0} z \right) + B \sinh \left( \frac{N \kappa}{f_0} z \right). \] (5)

Thus, we can regard this as the sum of two exponentials, decaying away from the lower and upper boundaries.

To close the problem, we need to invoke the upper and lower boundary conditions \( w' = 0 \). To do so, consider the thermodynamic equation, which for this Boussinesq system is just \( d\rho/dt = 0 \), which yields the linearized perturbation equation

\[ \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \rho' + v' \frac{\partial \rho_0}{\partial y} + w' \frac{\partial \rho_0}{\partial z} = 0. \]

Since \( \rho' = -(f_0 \rho_{00}/g) \partial \psi'/\partial z \), and \( \partial \rho_0/\partial y = f \Lambda \rho_{00}/g \), and \( w' = 0 \) on the boundaries, we have

\[ (U - c) \frac{d\Phi}{dz} - \Lambda \Phi = 0 \] (6)
on each boundary.

It is convenient here to define the length scale, \( L = ND/f_0 \), the *internal radius of deformation*. This compares with the external radius deformation, \( L_e = \sqrt{gD/f_0} \):

\[
L = \frac{ND}{f_0} = \frac{D}{f_0} \sqrt{\frac{g}{\rho_{00}} \left| \frac{d\rho_0}{dz} \right|} = \frac{D}{f_0} \sqrt{\frac{g}{D} \frac{\Delta\rho_0}{\rho_{00}}} = L_e \sqrt{\frac{\Delta\rho_0}{\rho_{00}}}.
\]

Then \( N\kappa z/f_0 = \kappa Lz/D \).

Now, applying (5) to each boundary condition (6) in turn, noting that \( U = \Lambda D/2, -\Lambda D/2 \), on the upper and lower boundaries respectively,

\[
\frac{\kappa L}{D} \left( \frac{\Lambda D}{2} - c \right) \left[ A \sinh \left( \frac{1}{2} \kappa L \right) + B \cosh \left( \frac{1}{2} \kappa L \right) \right] - \Lambda \left[ A \cosh \left( \frac{1}{2} \kappa L \right) + B \sinh \left( \frac{1}{2} \kappa L \right) \right] = 0
\]

\[
-\frac{\kappa L}{D} \left( \frac{\Lambda D}{2} + c \right) \left[ -A \sinh \left( \frac{1}{2} \kappa L \right) + B \cosh \left( \frac{1}{2} \kappa L \right) \right] - \Lambda \left[ A \cosh \left( \frac{1}{2} \kappa L \right) - B \sinh \left( \frac{1}{2} \kappa L \right) \right] = 0
\]

Eq. (7) represents an eigenvalue problem for \( c \). After a good deal of manipulation (see the Appendix for details) we find

\[
c = \pm \frac{\Lambda D}{\kappa L} \sqrt{\left[ \frac{\kappa L}{2} - \tanh \left( \frac{1}{2} \kappa L \right) \right] \left[ \frac{\kappa L}{2} - \coth \left( \frac{1}{2} \kappa L \right) \right]}.
\]

The function \((x - \tanh x)(x - \coth x)\) is plotted in Fig. 1. When \( x < 1.1997 \), the function is negative\(^1\), and \( c \) is then purely imaginary; when \( x > 1.1997 \), \( c \) is purely real. Note that since our solution (4) depends on time as \( \exp(-ikct) \), we have propagating waves, without growth or decay, for

\(^1\)In fact, the zero of the function occurs where \( x = \coth x \).
Figure 1: The function $y = (x - \tanh x)(x - \coth x)$.

$\text{Im}(c) = 0$, and growing\(^2\) waves for $\text{Im}(c) > 0$. Since, from (8), we have

$$\frac{c}{\Lambda D} = \pm \frac{1}{\kappa L} \sqrt{\left(\frac{\kappa L}{2} - \tanh \left(\frac{1}{2}\kappa L\right)\right) \left(\frac{\kappa L}{2} - \coth \left(\frac{1}{2}\kappa L\right)\right)}$$

(9)

we can plot $c/\Lambda D$ vs. $\kappa L$; this is shown in Fig. 2.

The gross characteristics of the solutions, therefore, depend solely on whether or not $\mu D = f_0 \kappa D/N$ exceeds the value $\gamma_0 = 2.3994$. For given $N$ and $D$, the long waves grow:

$$\kappa < \gamma_0 L^{-1}, \quad \text{Im}(c) \neq 0$$

$$\kappa > \gamma_0 L^{-1}, \quad \text{Im}(c) = 0$$

For very short waves, $\kappa L \gg 1$ and (8) gives us

$$c \to \pm \frac{1}{2} \Lambda D.$$

In this limit, the $\sinh$ and $\cosh$ functions just decay exponentially away from the boundaries (and have little amplitude at the opposite boundary).

\(^2\)Since (8) has two solutions of opposite signs, whenever there is a growing solution, there is also a corresponding decaying solution. But the growing solution with positive Im part is the interesting one.
These “Eady edge waves” (which are formally equivalent to Rossby waves, owing their existence to the temperature gradients there) are trapped at each boundary, and each is simply advected by the local flow. For smaller $\kappa$, the two boundary waves decay less rapidly with height and interact, slowing each other’s propagation. Eventually, when $\kappa$ exceeds the critical value, this interaction stalls the waves ($\text{Re}(c) \to 0$, so the waves propagate at the speed of the mid-level flow, which happens to be zero in this case) and they begin to reinforce each other, causing growth of the coupled boundary waves.

Even though $c$ depends on wavenumber only through its magnitude $\kappa$, the growth rate of growing waves, $\sigma = k \text{Im}(c)$, is a function of $k$ and $l$. Writing dimensionless wavenumbers $k' = kL$, $l' = lL$, $\kappa' = \sqrt{k'^2 + l'^2} = \kappa L$, we have

$$\frac{\sigma L}{\Lambda D} = k' \text{Im} \left( \frac{c}{\Lambda D} \right)$$

and so from (9) we have

$$\frac{\sigma N}{f_0\Lambda} = \pm \left( \frac{k'}{\kappa'} \right) \sqrt{\left[ \frac{\kappa'}{2} - \tanh \left( \frac{1}{2} \kappa' \right) \right] \left[ \frac{\kappa'}{2} - \coth \left( \frac{1}{2} \kappa' \right) \right]}.$$
The dependence of $\sigma N/f_0 \Lambda$ on $k'$ and $l'$ for the growing wave is plotted in Fig. 3. The maximum growth rate, $\sigma N/f_0 \Lambda = 0.31$, is found at $k = 1.61L^{-1}$, $l = 0.05$. Note that the growth rate $\sigma$ depends on the ratio $\Lambda/f_0 N$, and therefore, as one might expect, increases with increasing baroclinic shear $\Lambda$, but that the wavelength of the fastest growing wave is independent of $\Lambda$.

Is this instability relevant to the real world? In the midlatitude troposphere, $D \approx 10 \text{km}$, $N \approx 1.2 \times 10^{-2} \text{s}^{-1}$, $f_0 \approx 1.0 \times 10^{-4} \text{s}^{-1}$, and $\Lambda$ is typically $25 \text{ms}^{-1}/10 \text{km} \approx 2.5 \times 10^{-3} \text{s}^{-1}$. So the fastest growth rate is
$0.31 \times 2.5 \times 10^{-7}/1.2 \times 10^{-2} \simeq 6.5 \times 10^{-6} \text{s}^{-1}$, or an $\varepsilon$-folding time scale of $1.5 \times 10^6 \text{s} \simeq 1.8 \text{ days}$. This is comparable with what is seen in a strongly developing storm. The wavenumber of the fastest growing wave is $1.61 f_0/ND = 1.61 \times 10^{-4}/(120) \text{ m}^{-1} \simeq 1.34 \times 10^{-6} \text{ m}^{-1}$, giving a wavelength of $2\pi/k \simeq 4700 \text{ km}$. (At $45^\circ$, where a latitude circle measures 28000 km, this corresponds to zonal wavenumber 6.)

The longitude-height structure of the most rapidly growing mode is shown in Fig. 4. Note:

![Diagram](Holton Fig 8.10)

Figure 4: [Holton Fig 8.10]
1. The geopotential perturbation maximizes at the upper and lower boundaries

2. The geopotential perturbation tilts westward with height

3. \( w \) and \( T \) are positively correlated

4. Poleward flow (\( \partial \phi / \partial x > 0 \)) is positively correlated with \( T \)

### 5.3 Instability of zonal flows (QG): Energetics of the Eady problem

[Holton, Ch. 8 discusses energetics in a different situation. Lorenz (The Nature and Theory of the General Circulation of the Atmosphere, WMO, Geneva, 1967) discusses atmospheric energetics in some detail. When applied locally, details can be over-interpreted (Plumb, J. Atmos. Sci., 40, 1670-1688, 1983) but we avoid these pitfalls here by focusing exclusively on integrated budgets.]

#### 5.3.1 Available potential energy

For a Boussinseq fluid (as in the problem at hand) the potential energy (PE) of an elementary fluid volume \( dV = dx
dy
dz \) is \( \rho g z
dV \), so the total amount in the fluid is

\[
P = g \int \int \int \rho z
dx
dy
dz .
\]

Note that the origin of \( z \) is arbitrary: one is only ever interested in changes of PE, however, this is not a very useful expression for a fluid, since there are other constraints, such as conservation of total mass. Moreover, if the dynamics are adiabatic, density can be changed only by redistribution, and if the flow satisiﬁes QG assumptions then, to leading order in Rossby number, that redistribution takes place (approximately) horizontally. Given our governing QG equations

\[
\begin{align*}
D \dot{u} - f v &= - \frac{\partial \phi}{\partial x} \\
D \dot{v} + f u &= - \frac{\partial \phi}{\partial y} \\
\frac{d\rho}{dt} &= 0
\end{align*}
\]

(11)
Here we have subtracted the horizontally averaged geopotential $\rho_0^{-1} \rho_0(z)$ from $\phi$, so that
\[ \frac{\partial \tilde{\phi}}{\partial z} = -\frac{g}{\rho_0} (\rho - \rho_0(z)) = -\frac{g}{\rho_0} \tilde{\rho}, \]
where $\tilde{\rho}$ is the departure of density from its horizontal average. Taking $u \times$ the first of (11) plus $v \times$ the second gives
\[ D_g \left[ \frac{1}{2} (u^2 + v^2) \right] = -u \frac{\partial \tilde{\phi}}{\partial x} - v \frac{\partial \tilde{\phi}}{\partial y} \]
\[ = -\nabla \cdot \left( \mathbf{u} \tilde{\phi} \right) + \frac{\partial \tilde{\phi}}{\partial z} \]
\[ = -\nabla \cdot \left( \mathbf{u} \tilde{\phi} \right) - \frac{g}{\rho_0} w \tilde{\rho}. \quad (12) \]

But from the QG version of the density equation can be written as
\[ D_g \tilde{\rho} + w \frac{d\rho_0}{dz} = 0 \]
so that multiplying by $\tilde{\rho}$,
\[ D_g \left( \frac{1}{2} \tilde{\rho}^2 \right) + w \tilde{\rho} \frac{d\rho_0}{dz} = 0. \quad (13) \]

So (12) and (13) together give
\[ D_g \left[ \frac{1}{2} (u^2 + v^2) + \frac{1}{2} \left( \frac{g}{\rho_0} \right)^2 \tilde{\rho}^2 \right] = -\nabla \cdot \left( \mathbf{u} \tilde{\phi} \right). \quad (14) \]

where we have used $N^2 = -(g/\rho_0) d\rho_0/\rho_0$. Eq. (14) expresses conservation of energy in this system. The term on the RHS, the divergence of what is often (and sometimes misleadingly) referred to as the energy flux, obviously vanishes on integration if there is no flow through the boundaries. The first term inside the bracket on the left is clearly identifiable as the kinetic energy per unit mass; the second term, slightly less obviously, is a measure of the potential energy per unit mass, such that
\[ A = \frac{1}{2} g^2 \rho_0 \frac{d\rho_0}{dz}. \quad (15) \]
is referred to as the *available potential energy* (APE). It is clearly not the same thing as (10), but the difference,

\[
P - A = \int \int \int \left( \rho z - \frac{1}{2} \frac{g^2}{\rho_{00}} \frac{\rho^2}{N^2} \right) \, dx \, dy \, dz
\]

is clearly something that is *unavailable* for conversion to kinetic energy by adiabatic QG motions.

Note from the form of (15) that APE depends only on departures of density from its horizontal average, *i.e.*, on horizontal gradients of density. A system with no horizontal gradients of density has no APE: the PE of such a state cannot be changed by adiabatic QG motions. The minimum APE (and, by (14) the maximum KE) is achieved by reducing \( \rho^2 \) to zero, *i.e.*, by flattening the density surfaces.

### 5.3.2 Eddy energies and growth

Now let’s consider the eddies separately. As before, we define eddy fluctuations to be the departures from the zonally averaged state:

\[
\bar{a} = \frac{1}{L} \int_{0}^{L} a \, dx ; \quad a' = a - \bar{a} .
\]

Then the total KE is the sum of mean KE \((K_M)\) and eddy KE \((K_E)\) where

\[
K_M = \frac{1}{2} L \int \int \rho_{00} \bar{u}^2 \, dy \, dz ; \quad K_E = \frac{1}{2} L \int \int \rho_{00} \left( \bar{u}^2 + \bar{v}^2 \right) \, dy \, dz . \tag{16}
\]

Similarly, the mean and eddy APEs are

\[
A_M = \frac{1}{2} \frac{g^2 L}{\rho_{00}} \int \int \frac{\bar{\rho}^2}{N^2} \, dy \, dz ; \quad A_E = \frac{1}{2} \frac{g^2 L}{\rho_{00}} \int \int \frac{\bar{\rho}^2}{N^2} \, dy \, dz . \tag{17}
\]

Now, going back to our normal mode stability problem, an initially infinitesimal disturbance, which by (16) and (17) has infinitesimally small \( K_E \) and \( A_E \), grows exponentially with time. As it does so, both \( K_E \) and \( A_E \) must increase with time. Under what conditions can this happen?

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3 It can, of course, be changed by non-QG motions: *convective available potential energy* (CAPE) is a measure of how much PE can be released by vertical motions.

4 Note that the mean geostrophic \( v \) vanishes by definition: \( \bar{v} = f^{-1} \partial \bar{\phi} / \partial x = 0. \)
The perturbation equations, linearized about the zonal mean, are

\[
\begin{align*}
\frac{\partial u'}{\partial t} &+ \bar{u} \frac{\partial u'}{\partial x} - f v' = -\frac{\partial \phi'}{\partial x} \\
\frac{\partial v'}{\partial t} &+ \bar{u} \frac{\partial v'}{\partial x} + f u' = -\frac{\partial \phi'}{\partial y} \\
\frac{\partial \rho'}{\partial t} &+ \bar{u} \frac{\partial \rho'}{\partial x} + \bar{v} \frac{\partial \rho'}{\partial y} + \bar{w} \frac{\partial \rho'}{\partial z} = 0
\end{align*}
\]  

(18)

Take \(u'\times\) the first + \(v'\times\) the second, average, and manipulate as before:

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \left( \bar{u}'^2 + \bar{v}'^2 \right) \right] = - \nabla \cdot \left( \bar{u}' \bar{\phi}' \right) - \frac{g}{\rho_{00}} \bar{w}' \bar{\rho}' ,
\]

which integrates (assuming no flow through the boundaries) to give simply

\[
\frac{dK_E}{dt} = -gL \int \int \bar{w}' \bar{\rho}' \, dy \, dz .
\]  

(19)

Thus, eddy KE grows if, on average, dense fluid is sinking and buoyant fluid rising within the eddy field.

To get the eddy APE equation, take \(\rho'\times\) the third of (18) and average to give

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \bar{\rho}'^2 \right) + \bar{v}' \bar{\rho}' \frac{\partial \bar{\rho}}{\partial y} + \bar{w}' \bar{\rho}' \frac{\partial \bar{\rho}}{\partial z} = 0 .
\]

Then from (17) we have

\[
\frac{dA_e}{dt} = -\frac{g^2L}{\rho_{00}} \int \int \frac{1}{N^2} \bar{u}' \bar{\rho}' \cdot \nabla \bar{\rho} \, dy \, dz ,
\]  

(20)

where \(\bar{u}' \bar{\rho}' = (\bar{u}' \bar{\rho}', \bar{v}' \bar{\rho}', \bar{w}' \bar{\rho}')\) is a measure of the eddy mass flux. (Note that the zonal component is irrelevant as \(\bar{\rho}\) has no zonal gradient.) What (20) implies is just a little more subtle than (19): in order to eddy APE to grow, projection of the eddy mass flux onto the mean density gradient must be downgradient.

How the two conditions for growth can be satisfied simultaneously can be seen from Fig. 5. In the mean state, density increases downward and poleward. Growth of eddy KE requires, from (19) that \(\bar{w}' \bar{\rho}'\) be downward, which is actually against the mean vertical gradient (toward higher density)
in order that, simultaneously, the projection of the flux onto $\nabla \bar{\rho}$ be negative, the flux must lie within the “wedge of instability” bounded by the density surface and the horizontal, as illustrated by the shading. (Note that the wedge of instability is often described in terms of parcel displacements, which can be problematic: the pitfalls are avoided by phrasing it in terms of buoyancy fluxes, as done here.) So, although neither (19) nor (20) alone states this directly, together they prescribe that the eddy mass flux must have a latitudinal component directed toward the lower mean density. Thus, the eddy mass flux must be downward and equatorward.

In an atmospheric context, the same principle holds, though we have to talk about the structure of $\theta$ in $\rho$ coordinates. In this case, eddy growth requires that the eddy flux of potential temperature, $\mathbf{u}' \theta'$, must lie within the wedge defined by the contours of $\bar{\theta}$ and pressure surfaces, as shown in Fig. 6. Thus, the eddy heat flux must be upward and poleward in a developing baroclinic wave.

5.3.3 Implications for baroclinic wave structure

We saw that $w'$ and $T'$ are positively correlated in a growing Eady wave, as (19) implies they must, in order for eddy KE to grow. Something else evident from the structure is that the wave tilts westward with height. This
in fact follows directly from the result that, on average, $\overline{v'p'}$ be directed equatorward, i.e.,

$$f v' < 0.$$  

Since geostrophic and hydrostatic balance give us

$$f v' = \frac{\partial \phi'}{\partial x},$$

and

$$\rho' = -\frac{\rho_{00}}{g} \frac{\partial \phi'}{\partial z},$$

we have

$$f v' p' = -\frac{\rho_{00}}{g} \frac{\partial \phi'}{\partial x} \frac{\partial \phi'}{\partial z}.$$  

Now, if (Fig. 7) lines of constant $\phi'$ slope at an angle $\alpha$ to the horizontal, then

$$\tan \alpha = -\frac{\partial \phi'/\partial x}{\partial \phi'/\partial z},$$

whence

$$f v' p' = \frac{\rho_{00}}{g} \left(\frac{\partial \phi' / \partial x}{\partial \phi' / \partial z}\right)^2 \tan \alpha$$

14
Figure 7: Slope (in $y - z$ space) of a phase line.

and so the requirement that, on average, $\frac{\bar{v}'}{\bar{\rho}'} < 0$ implies that $\tan \alpha < 0$ on average: the phase lines (of constant $\phi'$) must slope westward with height in a developing disturbance.

An example (taken from Wallace and Hobbs) of such tilt leading to rapid development is shown in Fig. 8. On Nov 19, 12Z, a cold front lies across the eastern half of the continent, with a weak surface low (central pressure a little less than 1004 hPa) lying on the front, a little south of the Ohio valley. At 500 hPa, a major trough, propagating eastward, is situated some way to the west of the surface low. These two couple together and grow such that, within only 12 hrs, the surface low has deepened to 994 hPa, with the trough still to its west.

Note that the initial disturbance aloft was hardly infinitesimal (it rarely is): it was a large amplitude, pre-existing disturbance. So the initial growth was hardly normal-mode-like. Once the upper trough and developing surface low begin to interact, however, the disturbance gains more similarity to an Eady mode. When the disturbance matures, the upper and lower lows become aligned, and growth stops.
Figure 8: Rapid development, 1964 Nov 19-20. Note that the upper level trough is westward of the surface low.
Appendix to Section 5: Solution of (7).

To clean up the eqs. a bit, temporarily use the shorthand $S = \sinh \left( \frac{1}{2} \kappa L \right)$, $C = \cosh \left( \frac{1}{2} \kappa L \right)$. Then rewrite (7) as

$$A \left[ \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) S - \Lambda C \right] + B \left[ \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) C - \Lambda S \right] = 0$$

(21)

$$A \left[ \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) S - \Lambda C \right] + B \left[ -\kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) C + \Lambda S \right] = 0$$

Setting the determinant of coefficients to zero gives us

$$\left[ \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) S - \Lambda C \right] \left[ -\kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) C + \Lambda S \right]$$

$$- \left[ \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) C - \Lambda S \right] \left[ \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) S - \Lambda C \right] = 0$$

Reorganizing,

$$-\kappa^2 \frac{L^2}{D^2} \left[ \left( \Lambda \frac{D}{2} \right)^2 - c^2 \right] SC + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) C^2 + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) S^2 - \Lambda^2 SC$$

$$-\kappa^2 \frac{L^2}{D^2} \left[ \left( \Lambda \frac{D}{2} \right)^2 - c^2 \right] SC + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} + c \right) S^2 + \Lambda \kappa \frac{L}{D} \left( \Lambda \frac{D}{2} - c \right) C^2 - \Lambda^2 SC = 0 .$$

Then

$$2\kappa^2 \frac{L^2}{D^2} \left[ c^2 - \left( \Lambda \frac{D}{2} \right)^2 \right] SC - 2\Lambda^2 SC + \Lambda^2 \kappa L \left( C^2 + S^2 \right) = 0 .$$

Now, using the identities

$$\frac{\cosh^2 x + \sinh^2 x}{\cosh x \sinh x} = 2 \coth 2x$$

and

$$\coth 2x = \frac{1}{2} (\tanh x + \coth x)$$

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we arrive at

\[ c^2 = \left( \frac{\Lambda D}{2} \right)^2 - \frac{D^2 \Lambda^2}{L^2 \kappa^2} - \frac{\Lambda^2 D^2}{2 \kappa L} \left[ \tanh \left( \frac{1}{2} \kappa L \right) + \coth \left( \frac{1}{2} \kappa L \right) \right] \]

\[ = \left( \frac{\Lambda D}{\kappa L} \right)^2 \left[ \frac{\kappa^2 L^2}{4} - \frac{\kappa L}{2} \left( \tanh \left( \frac{1}{2} \kappa L \right) + \coth \left( \frac{1}{2} \kappa L \right) \right) \right] - 1 \]

\[ = \left( \frac{\Lambda D}{\kappa L} \right)^2 \left[ \frac{\kappa L}{2} - \tanh \left( \frac{1}{2} \kappa L \right) \right] \left[ \frac{\kappa L}{2} - \coth \left( \frac{1}{2} \kappa L \right) \right]. \quad (22) \]

This takes us directly to (8).