Notes on the governing equations for a compressible atmosphere

Pressure coordinates

For a compressible atmosphere, the equations of motion become a little more complicated than the incompressible case. The main issues are

1. the pressure gradient term \(-\rho^{-1}\nabla p\) in the equations of motion, which is nonlinear when \(\rho\) is a dependent variable, and

2. the continuity equation

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0
\]

(1)

in which the velocity is no longer nondivergent.

Things become much simpler, if we can make the hydrostatic approximation (which we will do for all large-scale motions), when pressure (or some function of pressure) is used instead of \(z\) as the vertical coordinate. For example, in the pressure gradient term, note that, with \(p = p(x, y, z, t)\) and \(z = z(x, y, p, t)\),

\[
\left( \frac{\partial p}{\partial x} \right)_p = 0 = \left( \frac{\partial p}{\partial x} \right)_z + \left( \frac{\partial z}{\partial x} \right)_p \left( \frac{\partial p}{\partial z} \right)_x
\]

Using hydrostatic balance, then,

\[
\left( \frac{\partial p}{\partial x} \right)_z = -g \rho \left( \frac{\partial z}{\partial x} \right)_p
\]

and similarly for the \(y\)--derivative. Accordingly, the pressure gradient term becomes the horizontal components of

\[
-\frac{1}{\rho} (\nabla p)_z = -g (\nabla z)_p = - (\nabla \phi)_p
\]

where \(\phi = gz\) is the geopotential. The nonlinearity has gone, provided we regard horizontal as being “at constant \(p\)” — i.e., we express the horizontal components of the momentum equations in \(p\) coordinates.
The continuity equation goes even more simply. Without going back to the elementary math of its derivation, note that (1), multiplied by the elemental volume $dx\,dy\,dz$, expresses

\textit{rate of change of mass of the element} + \textit{mass convergence into the element} = 0

The mass $dm = \rho\,dx\,dy\,dz$ of the element in a compressible fluid can change because density changes. Now, using hydrostatic balance,

\[ \rho\,dx\,dy\,dz = -g^{-1}\,dx\,dy\,dp . \]

So, in pressure coordinates, the “volume” element $dx\,dy\,dp$ does not change, and so the mass of the element is fixed (the “density” in $p$-coordinates is just $-g^{-1}$, a constant). Accordingly, the first term in (1) disappears, and the $p$-coordinate continuity equation is just

\[ \nabla_p \cdot \mathbf{u}_p = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 , \quad (2) \]

where $\omega = dp/dt$ is the “vertical velocity” in pressure coordinates: in $p$-coordinates, we recover the statement of non-divergent flow.

Finally, we just need to re-state the hydrostatic equation, which simply inverts to give

\[ \frac{\partial z}{\partial p} = -\frac{1}{g\rho} , \]

or

\[ \frac{\partial \phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p} . \]