

A: Dispersive and nondispersive waves

1.1 Background theory—nondispersive waves

1.1.1 Oscillations

Oscillations (*e.g.*, small amplitude oscillations of a simple pendulum) are often described by an equation of the form

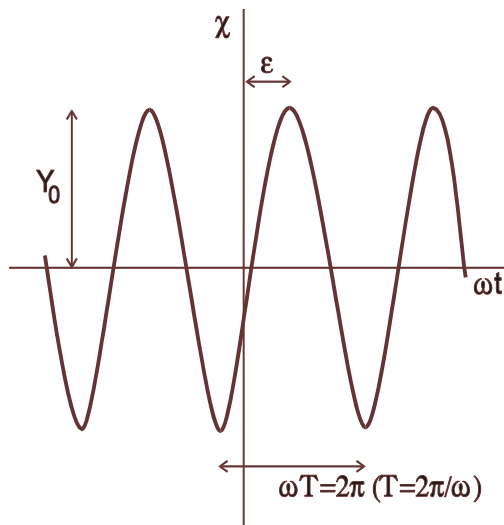


Figure 1: Characteristics of a simple oscillation.

$$\frac{d^2\chi}{dt^2} + \Omega^2\chi = 0 \quad (1)$$

where t is time and χ is some system variable (angular displacement in the case of the simple pendulum). (1) has solutions of the form

$$\chi(t) = \text{Re} (Y e^{i\omega t}) = Y_0 \cos(\omega t - \epsilon) \quad (2)$$

where frequency $\omega = \pm\Omega$, amplitude Y_0 and phase ϵ are real constants, and $Y = Y_0 e^{-i\epsilon}$ is a complex amplitude. Thus, as shown in Fig. 1, an oscillation is characterized by three constants: amplitude, frequency, and phase.

1.1.2 Nondispersive waves

Unlike such simple oscillations, waves are functions of both time and space. The simplest wave equation is of the form

$$\frac{\partial^2 \chi}{\partial t^2} - c_0^2 \frac{\partial^2 \chi}{\partial x^2} = 0 \quad (3)$$

where c_0 is some constant. This equation could represent the electric or magnetic fields *in vacuo*, with c_0 the speed of light; or pressure perturbations in a compressible fluid, with c_0 the sound speed, or waves on shallow water.

We can find solutions to (3) by *separating the variables*, writing

$$\chi(x, t) = \text{Re} [A(t)B(x)] .$$

More specifically, if we look for “wave-like” solutions for which $B(x) = e^{ikx}$, where k is a real wavenumber (so $2\pi/k$ is wavelength)¹, then $d^2B/dx^2 = -k^2B$, so

$$\frac{\partial^2 \chi}{\partial x^2} = \text{Re} \left[A(t) \frac{d^2 B}{dx^2}(x) \right] = -k^2 \text{Re} [A(t)B(x)] ,$$

and (3) becomes

$$\frac{d^2 A}{dt^2} + k^2 c_0^2 A = 0 .$$

This has solutions like

$$A = \chi_+ e^{-i\omega t} ; A = \chi_- e^{+i\omega t}$$

where χ_+ and χ_- are constant (complex) amplitudes and the frequency ω satisfies

$$\omega^2 = k^2 c_0^2 . \quad (4)$$

The full solution is

$$\chi(x, t) = \text{Re} [\chi_+ e^{i(kx - \omega t)} + \chi_- e^{i(kx + \omega t)}] . \quad (5)$$

Each of the two terms in (5) describes a **progressive wave** (Fig. 2):

- at any instant, it is just a sinusoidal wave disturbance, of **wavelength** $2\pi/k$

¹In general, any function of x can be expressed as a Fourier integral of such waves.

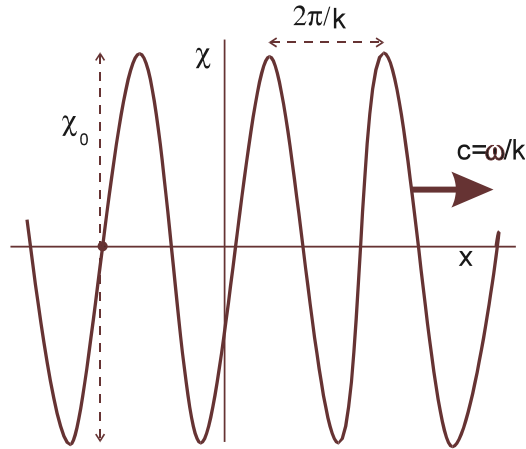


Figure 2: Characteristics of a progressive wave.

- at any fixed location $x = x_0$, it is just an oscillation of the form $De^{\pm i\omega t}$, where $D = \chi_{\pm} e^{ikx_0}$ is its complex amplitude, of **period** $T = 2\pi/\omega$
- it **propagates** with **phase speed** $c = \omega/k = \pm c_0$. Note from (5) that the first term of χ is constant along characteristics with $kx - \omega t = \text{constant}$, *i.e.*, $x = \frac{\omega}{k}t + \text{constant}$. The second term is constant along $x = -\frac{\omega}{k}t + \text{constant}$. Thus, (5) represents two solutions, propagating in opposite directions with phase speed c_0 .

Eq. (4) is the **dispersion relation** for the wave: for a given wavenumber k , it tells us the wave's frequency. This form is particularly simple, as shown in Fig. 3.

[Note that this is drawn for positive k only—we may define k positive, without loss of generality, as long as we do not try to constrain the sign of ω .] The phase speed $c = \omega/k = \pm c_0$; the waves can propagate in either direction.

These waves are **nondispersive**, *i.e.*, their phase speed is independent of wavenumber. Thus, all waves, of any wavenumber, propagate at the same speed (in either direction), which means that non-sinusoidal disturbances propagate *without change of shape*. In fact, any function

$$\chi(x, t) = F(x \pm c_0 t) \tag{6}$$

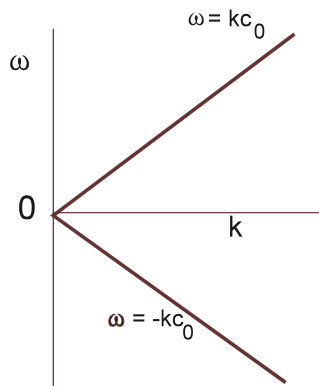


Figure 3: The dispersion relation for 1-D shallow water waves.

is a solution to (3)². Eq. (6) just describes any shape of disturbance, including a localized one, that propagates at speed c without changing its shape (Fig. 4).

1.1.3 Two-dimensional waves

In two dimensions (x, y) , (3) is replaced by

$$\frac{\partial^2 \chi}{\partial t^2} - c_0^2 \nabla^2 \chi = 0 \quad (7)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We can now look for **plane wave** solutions of the form

$$\chi(x, y, t) = \text{Re} [A_0 e^{i(kx+ly-\omega t)}] . \quad (8)$$

²To see this, note that if $X = x \pm c_0 t$, then $F = F(X)$ and the chain rule gives us

$$\begin{aligned} \frac{\partial \chi}{\partial x} &= \frac{\partial F}{\partial x} = \frac{dF}{dX} \frac{\partial X}{\partial x} = \frac{dF}{dX} ; \\ \frac{\partial^2 \chi}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{dF}{dX} \right) = \frac{\partial X}{\partial x} \frac{d^2 F}{dX^2} = \frac{d^2 F}{dX^2} ; \\ \frac{\partial \chi}{\partial t} &= \frac{\partial F}{\partial t} = \frac{dF}{dX} \frac{\partial X}{\partial t} = \pm c_0 \frac{dF}{dX} ; \\ \frac{\partial^2 \chi}{\partial t^2} &= \frac{\partial}{\partial t} \left(\pm c_0 \frac{dF}{dX} \right) = \pm c_0 \frac{\partial X}{\partial t} \frac{d^2 F}{dX^2} = c_0^2 \frac{d^2 F}{dX^2} . \end{aligned}$$

So, (3) is satisfied by (6).

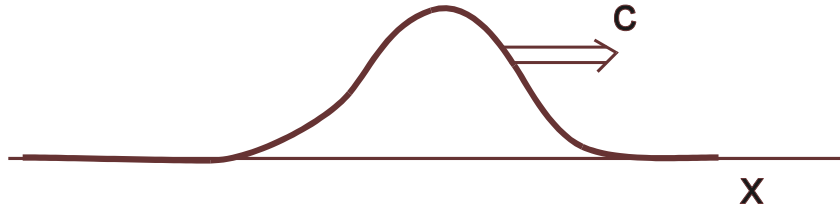


Figure 4: Nondispersive waves: arbitrary disturbances propagate without change of shape.

Then

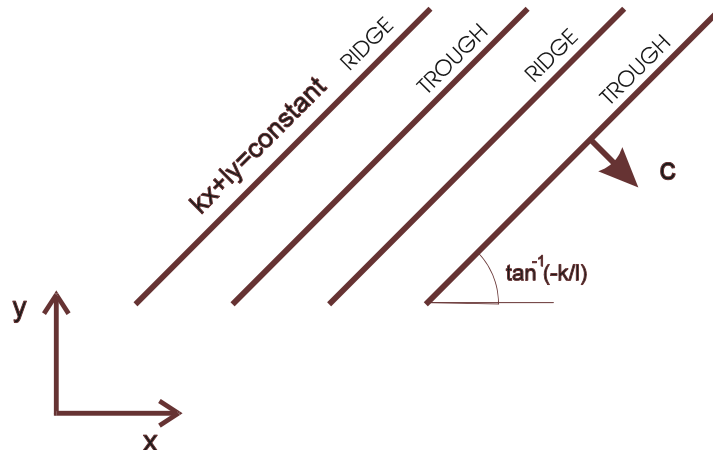
$$\nabla^2 \chi = -\kappa^2 \operatorname{Re} [A_0 e^{i(kx+ly-\omega t)}]$$

where $\kappa = \sqrt{k^2 + l^2}$ is the total wavenumber. Therefore, substituting into (7) gives the dispersion relation for this case

$$\omega^2 = \kappa^2 c_0^2. \quad (9)$$

[Note that the one-dimensional case we discussed above is just a special case of the two-dimensional problem, with $l = 0$.]

Eq. (8) describes a plane wave because χ is constant along lines of constant phase $kx + ly - \omega t = \text{constant}$, so at any instant in time, $kx + ly = \text{constant}$; see Fig. 1.1.3. The wave pattern moves at right angles to the phase



lines, with speed c .

Plane waves are a special, and particularly simple, form of 2-D waves. Exactly what shape the wavefronts have will in general depend on the geometry of the system and of the process that generated the wave. If the source is very localized (*e.g.*, a stone dropped into water), the wavefronts will be circular, as shown in Fig. 5. Note that, far from the source (in the dashed rectangle), the wavefronts will look almost plane.

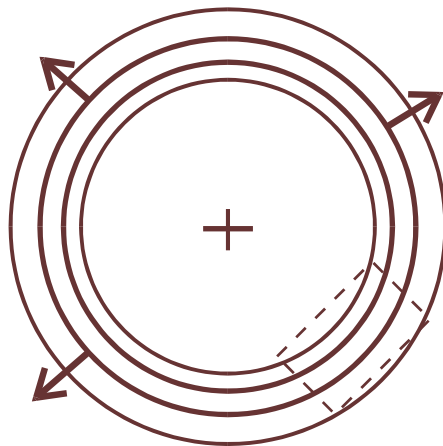


Figure 5: Circular wave fronts radiating from a localized source.

1.2 Background theory—dispersive waves

1.2.1 Dispersion

The general dispersion relation for the frequency ω of 1-D waves of wavenumber k can be written

$$\omega = \omega(k) \tag{10}$$

and the phase speed is

$$c = \frac{\omega(k)}{k} . \tag{11}$$

Clearly, c is independent of k for all k only if $\omega(k) = \text{constant} \times k$ —this is the nondispersive case we discussed earlier and, as we saw, it implies that all disturbances, including localized ones, propagate without change of shape. This can be thought of in terms of Fourier components. Any non-sinusoidal disturbance can be described a sum of components of different wavenumber;

if all these waves propagate at the same speed, so will the disturbance itself, and its shape will not change.

For many kinds of wave motion, however (including surface waves on deep water, internal gravity waves, and Rossby waves), c varies with k , in which case the different wavenumber components will have different speeds, a phenomenon known as **wave dispersion**. Therefore the way they interfere with one another will change with time—so the shape of the disturbance will change.

1.2.2 Group velocity

There is one particularly important aspect of dispersion, which concerns the way that modulations propagate on a wave train.

A monochromatic plane wave of the form

$$\chi(x, t) = \text{Re} [A_0 e^{i[k_0 x - \omega(k_0)t]}]$$

has a single wavenumber k . This is a special case of a more general form

$$\chi(x, t) = \text{Re} \left[\int_{-\infty}^{\infty} A(k) e^{i[kx - \omega(k)t]} dk \right] \quad (12)$$

where, in the plane wave case, the wavenumber spectrum has the simple form of a δ -function, $A(k) = A_0 \delta(k - k_0)$ [Fig. 6(a)]. The wave behaves in the way we have discussed, propagating with a speed $c = \omega(k_0)/k_0$. Consider

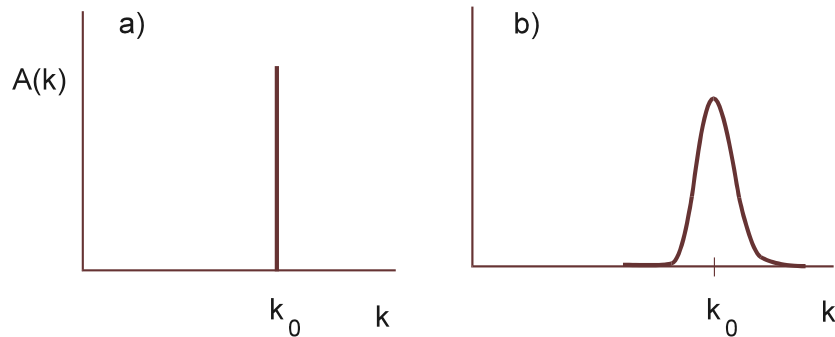


Figure 6: Wavenumber spectra for (a) a monochromatic wave, and (b) an almost-monochromatic wave.

now an almost monochromatic wave, with a narrow spectrum [Fig. 6(b)]. In

this case, the spectrum has finite but small width such that $A(k)$ is nonzero only for wavenumbers in the close vicinity of k_0 . Writing $k = k_0 + \delta k$, we can rewrite (12) as

$$\chi(x, t) = \text{Re} \int_{-\infty}^{\infty} A(k_0 + \delta k) e^{i[k_0 x - \omega(k_0)t]} e^{i[\delta k x - \delta \omega t]} d(\delta k) ,$$

where $\delta \omega = \omega(k_0 + \delta k) - \omega(k_0) \simeq (\partial \omega / \partial k) \delta k$. At $t = 0$, this is simply

$$\chi(x, t) = \text{Re} F(x) e^{ik_0 x}$$

where

$$F(x) = \int_{-\infty}^{\infty} A(k_0 + \delta k) e^{i \delta k x} d(\delta k)$$

is the modulating envelope of the wave train, which has carrier wavenumber k_0 , as illustrated in Fig. 7. Now, for $t > 0$, the wave packet behaves as

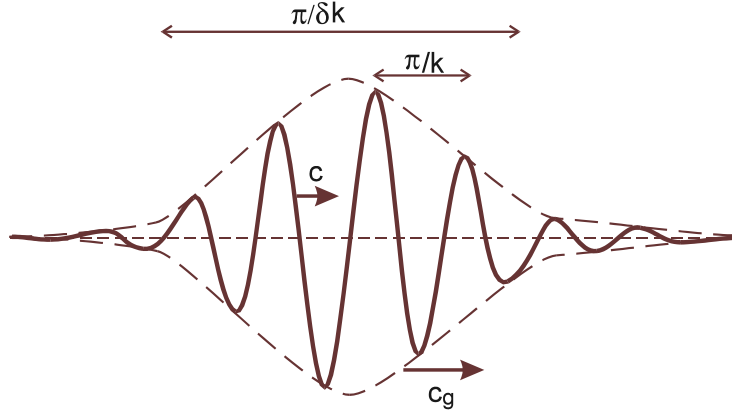


Figure 7: An almost-monochromatic wave packet, comprising many wavelengths. The phase of the carrier wave propagates at the phase speed, but the modulation envelope propagates at the group velocity.

$$\begin{aligned} \chi(x, t) &= \text{Re} \left\{ \left[\int_{-\infty}^{\infty} A(k_0 + \delta k) e^{i[\delta k(x - c_g t)]} d(\delta k) \right] e^{i[k_0(x - ct)]} \right\} \\ &= \text{Re} \left\{ F(x - c_g t) e^{i[k_0(x - ct)]} \right\} , \end{aligned} \quad (13)$$

where

$$c_g = \left. \frac{\partial \omega}{\partial k} \right|_{k=k_0} \quad (14)$$

is the **group velocity**. Thus, while the carrier wave propagates at the phase speed, the modulation envelope propagates at the group velocity. This is an important concept, as it is the latter velocity that governs the propagation of information, as we shall see.

Nondispersive waves have $\omega = c_0 k$, with constant phase speed c_0 , and so their group velocity is the same as their phase velocity. But the group velocity of dispersive waves differs from the phase speed, so in a wave packet like that shown in Fig. 7 the wave crests will move at a different speed than the envelope. If $c > c_g$ (which, as we shall see, is the case for deep water waves), new wave crests appear at the rear of the wave packet, move forward through the packet, and disappear at its leading edge. We shall see some examples of this below.

In general, it is easy to get a feel for both phase and group propagation graphically from the dispersion relation, as shown in Fig. 8. For any

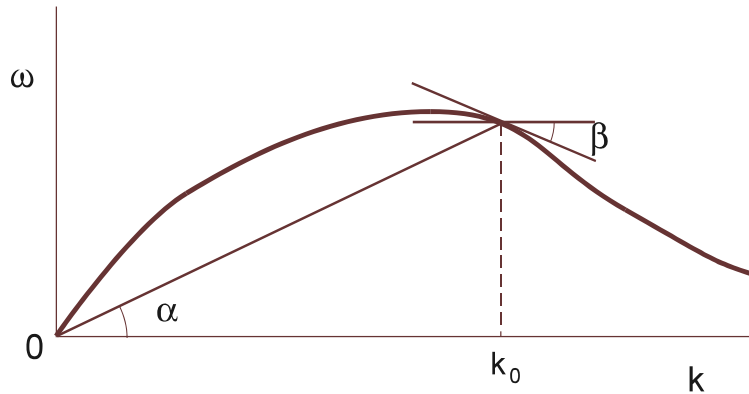


Figure 8: The dispersion relation $\omega(k)$. The phase velocity at $k = k_0$ is $\tan \alpha$, the group velocity $\tan \beta$.

wavenumber k , the phase velocity is just given by the slope α of a line joining the point (ω, k) to the origin, while the group velocity is given by the slope $\tan \beta$ of the tangent to the curve at (ω, k) .

1.2.3 Group velocity in multi-dimensional waves

For future reference, we note here that for 2- or 3-dimensional waves, the general dispersion relation is of the form

$$\omega = \omega(\mathbf{k}) \tag{15}$$

where $\mathbf{k} = (k, l, m)$ is the (2-D or 3-D) vector wavenumber. The phase velocity³ is

$$\mathbf{c} = \left(\frac{\omega}{k}, \frac{\omega}{l}, \frac{\omega}{m} \right), \tag{16}$$

and the group velocity

$$\mathbf{c}_g = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right). \tag{17}$$

As we shall see (and a comparison of (16) and (17) implies), not only may \mathbf{c}_g and \mathbf{c} be of different magnitudes, they may also be *in different directions*.

³The phase velocity is in fact not a vector, even though it has magnitude and direction. It does not transform like a vector under rotation—this stems from the fact that phase propagation has no meaning along the phase lines.